Shrapnel motion



Christopher James Poole
University College
University of Oxford

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Abstract

When a bomb explodes, the bomb-casing breaks up causing shrapnel fragments to scatter off with high velocity. Whilst study on penetration of the designated target has been addressed, the mechanics involved in the motion of the shrapnel have not been examined in great detail. This dissertation will address shrapnel motion using ideas from gasdynamics. The motivation for this problem comes from *QinetiQ* who, among other things, play an active role in modelling violent mechanics.

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Chapter 1

Introduction

1.1 Motivation

When a bomb explodes, fragments of the metal casing which initially surrounded the explosive mixture fly off with high velocity. The process of fragments forming is known as fragmentation, the metal fragments being *shrapnel*. Fragmentation warhead technology can be applied to many munition systems (e.g. detonation of bombs). This provides the motivation for understanding the dynamics behind fragmentation. If we can model how fragments form, fly off and penetrate their designated targets, then more efficient warheads can be designed so as to optimise destruction of their targets, whilst causing minimal unwanted damage. The study of fragmentation also has high significance in safety studies of containers that could possibly explode (e.g. pressurised containers containing explosive chemicals).

The problem of fragment penetration of a target has been studied in some detail, whereas there has been relatively little study of fragment formation (which occurs during the expansion of the casing) and subsequent motion of these fragments. Numerical simulations on fragmentation, known as *hydrocode* models, exist [4], although they cannot make accurate predictions of the fragmentation pattern owing to the underlying material models lacking sophistication.

There are three main types of fragment warhead:

- (i) Casings with fully preformed fragments on their outer surface (Fig. 1.1).
- (ii) Casing with known pre-existing fracture lines.
- (iii) So-called 'naturally fragmenting' warheads. These consist of a casing which cracks naturally as a result of the explosion.

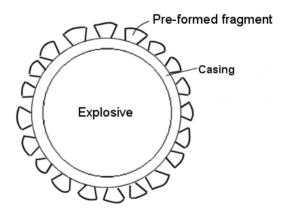


Figure 1.1: Bomb with preformed (man-made) fragments.

We are interested in modelling the flow of gas between the fragments in the early stages following detonation (Fig. 1.2). The ideas from the solution to these models can then hopefully be applied to all three types of casing. The mechanics of the break up of the casing are being studied in an MSc dissertation [7], drawing analogies with the notion of a shaped charge.

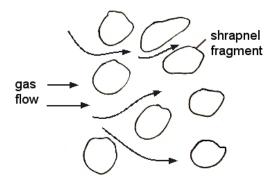


Figure 1.2: The shrapnel problem.

1.2 The shrapnel problem

A bomb basically consists of an explosive mixture surrounded by a metal casing. Two main bomb geometries will be discussed in this dissertation; spherical and cylindrical.

Fig. 1.3 shows a spherical casing [1]. The holes in the bottom and top show where the dynamite and fuse are inserted. Spherical bombs are usually detonated from the centre. The casing in this example is made from lead although modern casings are typically made from steel or sometimes aluminium.



Figure 1.3: Globe-shaped bomb casing from 1886.

The mechanism of the explosion is as follows. Initially, the explosive is detonated. Combustion occurs (starting at the ignition point), turning the explosive into high pressure gas. The gas pressure causes the casing to expand, at first elastically and then plastically [23]. The expansion also causes fracturing to occur in the casing. Cracks in the casing do not allow the gas to escape until the casing has expanded to roughly twice its original diameter [3, 12, 22]. After the casing has fractured, the high-velocity gas causes the casing to break up and form shrapnel. The gas then accelerates the shrapnel, flowing in the gaps between the fragments (Fig. 1.2). Hence the shrapnel flies off with high velocity. The gas flowing between the fragments and the shrapnel motion will be referred to as the *shrapnel problem*. An example of this process is shown in Fig. 1.4. The photographs show a cylindrical casing detonated at one end. This explodes, with a *detonation wave*¹ propagating down the cylinder.

G. I. Taylor's paper of 1941 [23] takes a mathematical and experimental look at one of the ideas mentioned above, namely a detonation starting at one end of a cylindrical casing as in Fig. 1.4. The different stages of the detonation are discussed. He then looks at the resultant flow and discusses fragment motion.

QinetiQ present further experimental evidence on a video which captures an explosion. As shown in Fig. 1.5, the motion occurs in four distinct phases. The first

 $^{^{1}\}mathrm{A}$ detonation is a form of shock wave where a chemical reaction occurs locally near the shock.

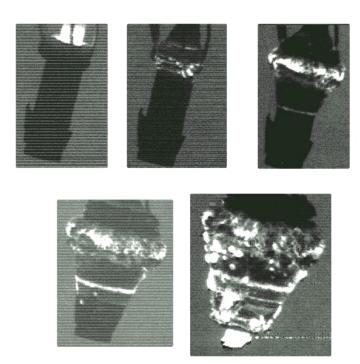


Figure 1.4: A sequence of photos showing bomb exploding, starting with the 'detonator initiation' and ending $20\mu s$ after detonation (from *QinetiQ* FP-23 hi-speed photography).

picture, (a), represents a small section of the casing before the combustion of the explosive has occurred.

At the next time step, (b), we can see a jet starting to form from between the preformed fragments. This is a consequence of the detonation wave having propagated from the explosive mixture.

These jets grow to roughly three times their previous length in stage (c). In this stage, the jets merge. Particle motion is then observed in (d). After this stage, we can see a jagged leading-edge on the outside of the casing, although the gases obscure the bulk of the particle motion. This indicates that the gas has rushed through the gaps between the fragments and is now moving ahead of the fragments with high velocity. Finally, everything on the video becomes unclear as a result of the gas obscuring the motion.

We can compare the video evidence to the photography in Fig. 1.4, although the latter is for a cylindrical casing. The same principles of shrapnel motion will apply for

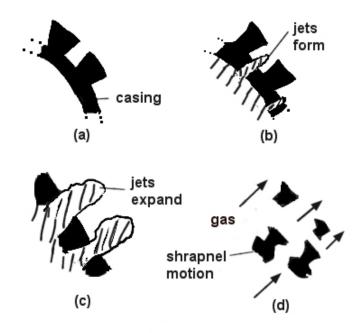


Figure 1.5: A sequence leading to shrapnel motion.

each type of casing, whether it is cylindrical or spherical with pre-formed or natural fragmentation.

1.3 Plan of dissertation

The background material on the subject is so sparse that the literature review will be integrated into the following chapters.

In chapter 2, where we discuss fluid flow, we will open with a brief background to incompressible flows. This will include a basic look at the equations and ideas governing such flows, including Navier-Stokes equations, Bernoulli's theorem, irrotational flow and vorticity. We will then move on to boundary layers, boundary layer separation and turbulence. These ideas will be applied to a nozzle with slowly-varying cross-sectional area, introducing the concept of a head loss. Pipe flows, inlets, orifices and screens will be reviewed, discussing head losses in each case. The second half of this chapter will be devoted to compressible flows. We will take a look at conservation laws, Rankine-Hugoniot conditions and some thermodynamics. We will complete the section on compressible flow background by looking at simple wave flow. These ideas will be used in a discussion of the shock tube and possible 'jump' conditions. We will then consider choked flow, before trying to generalise the specific situations considered for incompressible flow to end the chapter.

In chapter 3, we will attempt to bring these ideas together and set up some models of shrapnel motion. The first approach will model shrapnel as a piston expanding into a vacuum. This idea will be made more sophisticated in the second approach where we allow the piston to be porous. We will attempt to develop models for the incompressible case first, in order to gain intuition, and then go on to the compressible cases. The final model will use ideas from the theory of rock-blasting and consider how they might be applied to the shrapnel problem. We will complete this chapter by comparing the various models.

In chapter 4, we will give an overview of the dissertation and discuss possible modifications and future work to be carried out on the subject.

Chapter 2

Background Fluid Mechanics

We need to establish a detailed background on fluid mechanics to give us the infrastructure to develop a model on shrapnel motion. We aim to model the shrapnel problem for compressible gases. To gain intuition, we will firstly look at incompressible flow and then transfer ideas across to compressible flow. The simplest idea would be to consider these flows for 1-dimensional problems. This motivates the following sections.

2.1 Basic ideas about incompressible flow

A flow is *incompressible* if the density ρ of the fluid is constant everywhere. This section is devoted to a study of such flows, starting with a few preliminaries.

A good starting place for discussing incompressible flows is with the Navier-Stokes equations, which are well documented in any good fluid mechanics textbook [2]:

$$\rho \left(\frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \right) \boldsymbol{u} = -\nabla p + \mu \boldsymbol{\nabla}^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$
(2.1)

$$\nabla . u = 0 \tag{2.2}$$

where ρ is the fluid density, \boldsymbol{u} the fluid velocity, p the pressure and μ is the viscosity, which is assumed constant. We have neglected gravity in (2.1). These equations will hold throughout the fluid. The first of the equations (2.1), which is clearly nonlinear, can be derived by looking at conservation of momentum for a material volume of the fluid. The second of the two equations (2.2) is conservation of mass. This is easily derived by considering fluid flux across a volume V.

An *inviscid* fluid flow is one where viscous effects of the fluid are neglected. Hence (2.1) reduces to

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{u} = -\frac{1}{\rho} \nabla p. \tag{2.3}$$

This is known as Euler's equation. Assuming steady flow $(\frac{\partial}{\partial t} = 0)$ and using vector identities, this can be written as

$$(\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} = -\nabla(\frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2). \tag{2.4}$$

A streamline is a curve which has the same direction as \boldsymbol{u} at any time (Fig. 2.1). Taking the dot product of (2.4) with \boldsymbol{u} we obtain

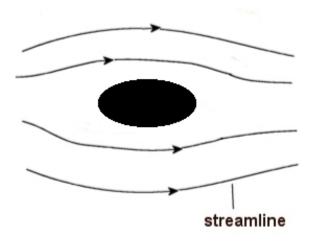


Figure 2.1: Streamlines.

$$(\boldsymbol{u}.\boldsymbol{\nabla})(\frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2) = 0. \tag{2.5}$$

Hence for steady flow,

$$(\frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2) = \text{constant on a streamline.}$$
 (2.6)

This is Bernoulli's equation.

We now introduce the *vorticity* of the fluid, defined by

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \boldsymbol{u}. \tag{2.7}$$

It can be shown [2] by considering the *circulation* $\Gamma = \int_C \boldsymbol{u} \cdot d\boldsymbol{x}$ around a closed curve C that if $\boldsymbol{\omega} = \boldsymbol{0}$ at t = 0, then $\boldsymbol{\omega} = \boldsymbol{0}$ for all time. This is the *Cauchy-Lagrange theorem*.

A flow with $\omega = 0$ is defined as *irrotational flow*. A special case of Bernoulli's equation arises in irrotational flow. Clearly, (2.4) reduces to

$$\nabla(\frac{p}{\rho} + \frac{1}{2}\mathbf{u}^2) = 0, \tag{2.8}$$

which means that the quantity $(\frac{p}{\rho} + \frac{1}{2}u^2)$ is constant throughout the fluid.

2.1.1 Boundary layers and turbulence

If we used the inviscid theory outlined above for flow past a rigid body, the fluid which comes in contact with the object would just slip past the object. This may give a good idea of the flow in general, but is in fact incorrect if we look more closely. We can nondimensionalise (2.1) in the form [18]

$$\left(\frac{\partial}{\partial t} + (\boldsymbol{u}.\boldsymbol{\nabla})\right)\boldsymbol{u} = -\nabla p + \frac{1}{Re}\boldsymbol{\nabla}^2\boldsymbol{u}$$
(2.9)

where

$$Re = \frac{UL}{\nu} \tag{2.10}$$

is the Reynolds number (nondimensional), U, L typical velocity and length scales and $\nu = \frac{\mu}{\rho}$ the kinematic viscosity. \boldsymbol{u} , $\boldsymbol{\nabla}$, p and t in (2.9) are now nondimensional. For $\frac{1}{Re} \ll 1$, we have a boundary layer. This is a thin layer of the fluid in which the fluid velocity changes rapidly from zero on the object to match in with the main fluid flow outside this layer. The mathematics can be demonstrated by using an ordinary differential equation¹:

$$\epsilon \frac{du}{dt} - u = -1,\tag{2.11}$$

with boundary condition

$$u(0) = 0, (2.12)$$

 $\epsilon \ll 1$. Here, the parameter ϵ is like $\frac{1}{Re}$ (very small). Naively neglecting the ϵ term leads to the solution u=1. This does not satisfy the boundary condition imposed. The remedy is to rescale x so that the ϵ term was of importance (e.g. $x=\epsilon X$, u(x)=U(X), then solve for U(X)). Our boundary layer, where rapid changes occur, is near x=0, of thickness ϵ . The same idea holds for the fluid - if we get sufficiently close to the boundary of the body, we have a boundary layer where the viscous effects will become important. An analysis of the boundary layer is possible by rescaling u and p. Suitable scalings involving Re for different flow regimes allow us to get good approximations in the boundary layers and in the main flow.

Clearly, the introduction of viscosity would not affect the incompressibility equation (2.2). It does mean, however, that we cannot use Euler's equation and must use the full Navier-Stokes equation (2.1). We use the nondimensional version (2.9), assuming steady flow $(\frac{\partial}{\partial t} = 0)$. One component of our equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{2.13}$$

¹A singular perturbation problem.

In the boundary layer, the viscous forces become more important, hence we wish to balance them with the pressure gradient (as we are looking at the shear in the x direction). We rescale [18]

$$y = \frac{Y}{\sqrt{Re}} \tag{2.14}$$

and hence, by incompressibility (2.2),

$$v = \frac{V}{\sqrt{Re}} \tag{2.15}$$

to obtain the boundary layer equation

$$\left(u\frac{\partial}{\partial x} + V\frac{\partial}{\partial Y}\right)u = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial Y^2}.$$
 (2.16)

Note that we almost have Euler's equation, apart from now we have a $\frac{\partial^2 u}{\partial Y^2}$ term which is of the same order as the other terms in the boundary layer. This viscous term means that we get different solutions in the boundary layer than in the main flow.

Boundary layers can be laminar or turbulent, but either can separate from the boundary under certain circumstances. *Boundary layer separation* can cause the whole flow of a low-viscosity fluid to change completely. Adverse pressure gradients

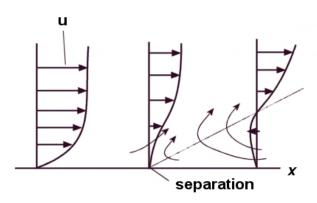


Figure 2.2: Boundary layer separation - velocity profile.

in the outer flow cause the flow to reverse near the boundary (Fig. 2.2) and this is observed to always lead to separation. The separation can lead to the formation of wakes behind the body (Fig. 2.3). The flow profile of Fig. 2.2 is slightly unrealistic in practice and we are more likely to get *triple deck* flow near the separation point.

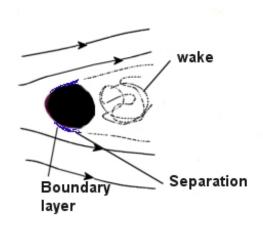


Figure 2.3: Boundary layer separation.

Details are beyond the scope of this dissertation, though can be found in Ockendon & Ockendon [18].

Separation can lead to turbulence in the main body of the flow. Turbulence is difficult to define. An area of turbulent flow will consist of vortices and eddies of many shapes and sizes, with the fluid particles following irregular paths. Turbulent flow is the opposite of laminar flow and is most easily generated from a strong shear. Two well-known examples of turbulence occurring owing to instabilities are Rayleigh-Taylor instability (denser fluid on top of a less dense one) and Kelvin-Helmholtz instability (layered fluids, with the upper layer of lower density flowing over the lower layer) [2].

2.1.2 Flow through a nozzle

(i) Inviscid theory

The first flow we shall consider is flow through a nozzle, shown in Fig. 2.4

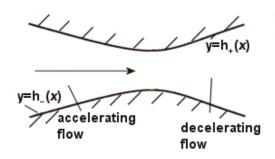


Figure 2.4: Flow through a nozzle.

As shown, neglecting viscosity, the flow will accelerate through the contracting part of the nozzle and decelerate as the nozzle expands. Considering the nozzle as a 2-dimensional model, we wish to obtain a slowly-varying solution of Euler's equation (2.3) with (inviscid) boundary conditions $\boldsymbol{u}.\boldsymbol{n} = 0$ holding on the nozzle boundary, \boldsymbol{n} being the normal to the nozzle. Denote upper and lower edges of the nozzle by $y = h_+(x)$ and $y = h_-(x)$ respectively. Writing the fluid velocity $\boldsymbol{u} = (u, v)$ in the usual manner, the boundary condition is

$$v = uh'_{\pm} \text{ on } y = h_{\pm}.$$
 (2.17)

We wish to solve

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.18}$$

with boundary conditions, so integrating between h_{-} and h_{+} with respect to y and using the boundary conditions we obtain

$$\frac{\partial}{\partial x}(u(h_+ - h_-)) = 0. \tag{2.19}$$

Rewriting the difference in heights $(h_+ - h_-)$ as A_α for $\alpha = 1, 2$, and taking a control volume of fluid (Fig. 2.5) we obtain conservation of mass in the form

$$A_1 u_1 = A_2 u_2 = Q, (2.20)$$

where Q is constant. Note that we can also apply the same idea to a 3-dimensional problem.

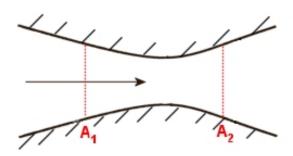


Figure 2.5: Flow through a nozzle (control volume).

We can obtain another control volume equation by observing that Bernoulli's equation (2.6) will hold on the centre streamline (laminar flow). In terms of our control volume subscripts, this says

$$p_1 + \frac{1}{2}\rho u_1^2 = p_2 + \frac{1}{2}\rho u_2^2. \tag{2.21}$$

A final balance is obtained by integrating

$$\nabla p + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = 0 \tag{2.22}$$

over the control volume region, V, with boundary S. This gives

$$\int_{S} (p\mathbf{n} + \rho \mathbf{u}(\mathbf{u}.\mathbf{n}))dS = 0.$$
 (2.23)

Hence the total force on the control volume is equal to the momentum flux from S. As the nozzle is assumed to be slowly varying, however, we know $\mathbf{u} \cdot \mathbf{n} = \mathbf{0}$ on $y = h_{\pm}$. Including the drag of the nozzle, this becomes

$$(p_1 + \rho u_1^2) - (p_2 + \rho u_2^2) = \text{Drag on nozzle.}$$
 (2.24)

(ii) Viscous effects, $Re \gg 1$

Experimentally, however, we sometimes obtain different results than the above. In particular, we note that Bernoulli is violated and

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + h_s, \tag{2.25}$$

where h_s is the *head loss*. The size and importance of this term depends very much on the geometry of the model. This can be rewritten as

$$h_s = \frac{p_1 - p_2}{\rho g} + \frac{u_1^2 - u_2^2}{2g}. (2.26)$$

We need to reconcile this equation with the nozzle flow. Hence we introduce a viscosity term, though still remembering that our flow is not very viscous (so $Re \gg 1$).

In the case of the nozzle, boundary layers form on the edge surface in the section of decelerating flow as shown in Fig. 2.6. These layers can separate as a result of the adverse pressure gradient on the edge of the expanding part, leading to turbulence in the bulk flow and hence causing the head loss term h_s in (2.25).

In general, the loss coefficient (or loss factor) is defined in terms of h_s as [5]

$$h_s = K \frac{u_2^2}{2g} \tag{2.27}$$

for some constant K, hence

$$\left(\frac{p_1}{\rho} + \frac{1}{2}u_1^2\right) - \left(\frac{p_2}{\rho} + \frac{1}{2}u_2^2\right) = \frac{1}{2}Ku_2^2. \tag{2.28}$$

Although we can write the loss in terms of the upstream velocity by (2.20), we choose to use the downstream velocity as the turbulent flow leading to the loss is downstream.

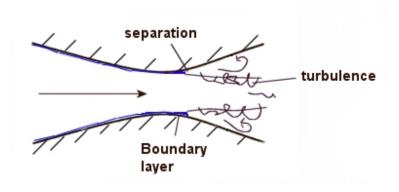


Figure 2.6: Boundary layer separation and turbulence in a nozzle.

2.1.3 Other pipe flows

(i) Gentle Contraction

A gentle contraction as shown in Fig. 2.7 is perhaps one of the easiest cases to look at. Mass conservation in the form (2.20) will obviously hold for the contracting pipe, with control volume boundaries denoted by 1 and 2. Also, as the pipe is contracting gently, there will not be any significant flow separation at the walls of the pipe. Hence we will not get any significant departure from inviscid flow and so we expect (2.21) to remain true. This is experimentally confirmed to be the case in engineering literature [5]. The only error we might get would arise from friction in (2.23).

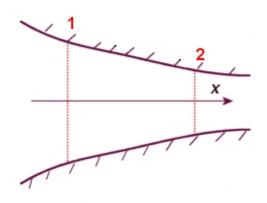


Figure 2.7: A pipe with a gentle contraction.

(ii) Gentle Expansion

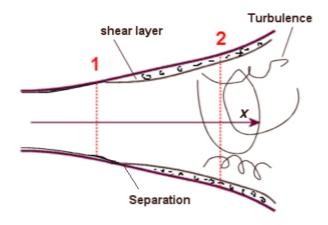


Figure 2.8: A pipe with a gentle expansion.

Fig. 2.8 shows a gentle pipe expansion. Mass conservation will hold in the form (2.20). The separation of the boundary layers leads to some turbulence in the middle of the pipe, disrupting the laminar flow. This will lead to a head loss. Thus (2.28) will hold for some K. The size of K will not be very large in this case, as the pipe cross-sectional area is only slowly varying. This implies that the size of h_s will not be very large compared to the other terms in (2.25). Analysis on curvature in pipes [21] suggests that this value will be negligible ($K \sim 0.04$ for a smooth bend), and hence we can realistically apply (2.21).

(iii) Sudden contraction

Fig. 2.9 shows a typical pipe contraction. 1, c and 2 refer to control volume subscripts which we shall refer to in the following text. The flow is divided into two regions [17], one of laminar flow and one where turbulence occurs. Firstly, the fluid accelerates through the contraction. The laminar structure of the flow is maintained (no turbulence) in this part of the flow.

Separation of shear layers then starts to occur. A vena contracta, with cross-sectional area A_c , forms as shown. This is the minimum area through which the fluid flows (like a neck). After this area, we effectively have an expansion. Shear layers separate, leading to turbulence and hence a head loss h_s and a loss factor K. Define the contraction coefficient by

$$C_c = \frac{A_c}{A_2}. (2.29)$$

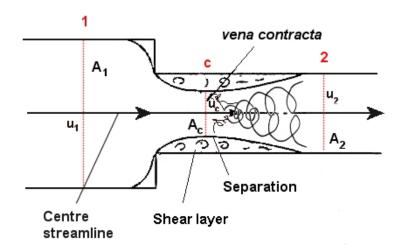


Figure 2.9: Flow through a pipe with a sudden contraction.

For a ratio of $\frac{A_2}{A_1} = 0.1$, then it is found experimentally that [17] $C_c \sim 0.62$. This leads to $K \approx 0.37$. A range of values are published in [24], shown in the following table. Note the low value of K for a high ratio of $\frac{A_2}{A_1}$. This reinforces the argument for a pipe with a gentle contraction that K can almost be neglected.

A_2/A_1	K
0.1	0.37
0.2	0.35
0.3	0.32
0.4	0.27
0.5	0.22
0.6	0.17
0.7	0.10
0.8	0.06
0.9	0.02
1.0	0

For a pipe with no drag, it is experimentally confirmed [5, 13, 17] that

$$K = \left(\frac{A_2}{A_c} - 1\right)^2 = \left(\frac{1}{C_c} - 1\right)^2 \tag{2.30}$$

is a good approximation for K.

(iv) Sudden expansion

We expect a sudden expansion (Fig. 2.10) to be more dramatic than the gentle expansion. Once again, we use our general principles and see at once that conservation of mass (2.20) must hold. We also have shear layers separating, causing turbulence which will stop the incoming flow from remaining laminar. Thus we expect there to be a significant head loss h_s and (2.28) will hold.

An experimental form for K is given by [5, 13, 17] as

$$K = \left(\frac{A_2}{A_1} - 1\right)^2. \tag{2.31}$$

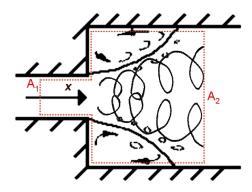


Figure 2.10: Flow through a pipe with a sudden expansion.

2.1.4 Inlet flow

Consider flow from a (relatively) large vessel into a small opening in the wall of the vessel. This opening is an *inlet* (Fig. 2.11). We are interested in finding loss coefficients for inlets as we can think of the gas flowing into the inlet as gas flowing between shrapnel fragments. The following arguments apply to stationary inlets. The ideas, however, will still follow for shrapnel (which moves) as the shrapnel will not be moving as fast as the high-speed gas.

We obtain values of the loss coefficient K experimentally [17]. It is found that they are strongly dependent on the geometry of the inlet. For the purpose of clarity of the Fig. 2.11, turbulent motion resultant from shear layer separation is not drawn on the diagrams, though it should be stressed again that this is the reason for Bernoulli's equation not holding, hence the loss factor K. Note that the strength of the separation force on shear layers will correlate with the effect (and amount) of head loss.

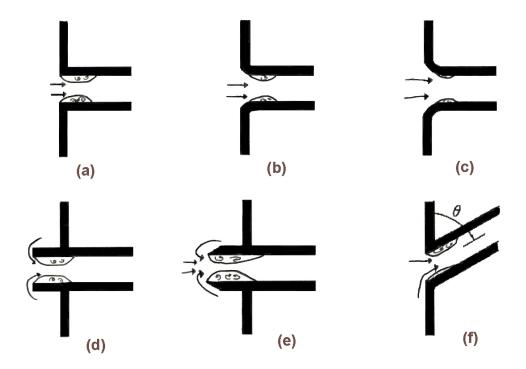


Figure 2.11: Different inlet geometries.

Firstly, in (a) we have what could be described as the simplest type of inlet. The corners of the inlet are sharp, which promotes the shear layers to separate. Hence we have the relatively high value of K = 0.50.

Inlet (b) has the corners on the entrance to the inlet slightly rounded. This will decrease the potential for shear layer separation as it is more similar to the case of a gently contracting pipe, through which the flow accelerates. In this case, K = 0.25.

Inlet (c) is a more extreme case of (b). The curvature of the entrance to the inlet ensures that the shear layers have little scope for separation, hence the very small range of values of K = 0.005 - 0.06. This experimental evidence backs up the ideas of neglecting the head loss in the case of the gentle contraction.

Note the difference in (a) and (d). In (d), the fluid can reverse (as the arrows indicate) to enter the pipe. Shear layers are built up earlier in the flow, and separate to cause more turbulence than in (a). This leads to a higher loss factor K = 0.56.

Inlet (e) reflects the latter idea. The walls of the inlet are shaped to allow more flow to enter and this leads to a very high level of shear layer separation, causing much more turbulence, hence the range $1.3 \le K \le 3.0$.

Finally, (f) shows an inlet at an angle θ . $\theta = 0$ will correspond to case (a). The angle has the effect of increasing the shear layers on one edge (top edge as

shown in the diagram) and decreasing it on the other. K is given empirically by $K = (0.5 + 0.3\cos\theta + 0.2\cos^2\theta)$.

Depending on the Reynolds number, the flow may or may nor become approximately laminar sufficiently far downstream. Hence, in addition to the experimental values given above, additional losses in the pipe will occur downstream before we get laminar flow (if it does occur), which is usually in about the first 30 pipe diameters. These losses often add about 0.1 to the loss coefficient K.

2.1.5 Orifices

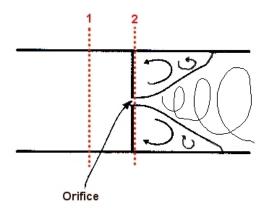


Figure 2.12: Flow through an orifice.

An orifice (Fig. 2.12) is a plate with a small, sharp-edged hole, placed in a flowing stream [10]. The motivation for considering an orifice is similar to the motivation for inlets. Gas flow past fragments in a casing will pass through small gaps, which we model here as a fixed orifice. Once again, even though the shrapnel will be moving, it will not have a velocity as high as the high-speed gas, hence we can gain insight by looking at the fixed orifice.

As the flow passes through the orifice, we will very quickly switch from a contraction to a very large expansion. Shear layers separate, leading to turbulence and a head loss.

Sharp-edged orifices with area ratios $\frac{A_2}{A_1}$ from 0.05 to 0.70 fit in well at high orifice Reynolds numbers with the relation [10]

$$K = 1.6 \left(1 - \left(\frac{A_2}{A_1} \right)^2 \right). \tag{2.32}$$

In this case, we can (if we wish) substitute K back into (2.28), rearrange and get

$$p_1 + 1.3\rho u_1^2 = p_2 + 1.3\rho u_2^2. (2.33)$$

i.e.

$$[p+1.3\rho u^2]_1^2 = 0. (2.34)$$

2.1.6 Screens

In the shrapnel problem, we will need to consider gas flow around many small fragments. This motivates looking at a *screen*.

A screen (Fig. 2.13) can be defined [8] as a collection of elements which forms a permeable sheet. This sheet is relatively thin in the direction of the flow through it. The gas flow accelerates through the gaps in the screen, forming jets of high velocity behind the holes. "Wakes" will form between the holes on the downstream side of the screen. These are areas of the fluid where the velocity is lower than in the jets. We will have eddies and turbulence in the wakes. Once again, further downstream, depending on the Reynolds number, the jets will mix with the wakes, eventually leading to uniform flow. It is found experimentally that this mixing process provides the main contribution to the pressure losses across the screen.

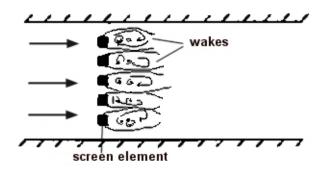


Figure 2.13: Flow past a screen.

It is found by Miller [16] that a number of geometrical parameters of the screen are not important, given that the separation and reattachment (where the jets and wakes come back together) of the flow are the same for each element of the screen. This means that we may effectively treat the screen as an array of pipes. For example, loss coefficients for orifice plates and perforated plates are similar if ratios of total orifice area to pipe area are the same (Fig. 2.14).

Also, for the same ratio of orifice to pipe area, the total *vena contracta* area is similar irrespective of whether there are single or multiple orifices. However, for a single orifice, the length of the outlet pipe required for full static pressure recovery greatly exceeds the length required for multiple orifices.

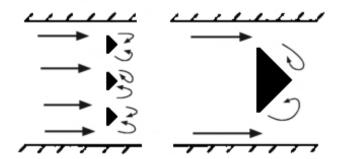


Figure 2.14: Two screens with similar geometrical ratios leading to similar pressure losses across the screen.

An important parameter when discussing screens is the *solidity* of the screen, defined as the ratio

$$S = \frac{\text{Blockage area of screen}}{\text{Area of pipe or passage}}.$$
 (2.35)

If we can obtain an expression for the loss K in terms of S, then we should have be able to get good estimates for K for different screens. Experimentally [16], values for round wire screens with the Reynolds number Re > 500 have been obtained, and are shown in Fig. 2.15.

Much investigation has been done on drag forces on screens owing to uses they have in the aerodynamics industry (e.g. in engines). One hypothesis [5] for the drag force D across the screen is

$$D = C_d A_{blocked} \left(\frac{\rho u^2}{2g}\right) \tag{2.36}$$

where C_d is the drag coefficient of the screen and $A_{blocked}$ the total blockage area of the screen. This relationship is generally found to be true (and the notion of a drag coefficient is not specific to screens.) The drag coefficient will be dependent on the geometry of the screen. It will also depend on the nature of the wakes which form downstream of the screen elements. Consider if the wakes are 'closed' (that is, they terminate and the flow becomes approximately laminar further downstream). The drag will depend on whether the wakes are stagnant (hence at stagnation pressure) or whether there are eddies in the wakes. If the latter holds, we can find (experimentally) a value for C_d which will depend on the screen element size and geometry. If the flow in the wakes is stagnant, the drag force will be as if the screen element and wake are one fixed body, and the drag will act around the boundary of the combined screen element and closed wake (as the pressure acting on the screen element from behind is negligible to that acting on the front).

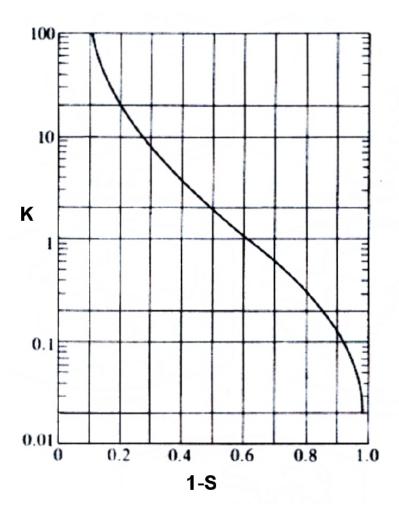


Figure 2.15: Loss factor K plotted against 1 - S [16].

Relationships between C_d and Re are known for low Reynolds number situations (Re < 5) [6]. This does not interest us much as we will be considering high Reynolds number flow.

2.2 Compressible flow

Up until now, we have only considered the fluid as being incompressible. This was done for simplicity. The shrapnel problem, however, will involve compressible flow. Hence, it will be more realistic to consider compressible flows in different geometries. A marked contrast between incompressible and compressible flows is that we can have shock waves forming in compressible flow, with *Rankine-Hugoniot* conditions holding across the shocks.

2.2.1 Shock waves

Consider a curve C across which some gasdynamic quantity (such as density) has discontinuities. The usual system of first order conservation laws cannot be interpreted over this curve as the first derivatives are not well-defined here. However, by stitching together classical solutions² across this curve, we can describe finite discontinuities (jumps) in the solution. These finite discontinuities in the solution are known as *shocks*. From a mathematical viewpoint, we note that by introducing shocks, we can avoid the solution from being multi-valued.

2.2.2 Rankine-Hugoniot jump conditions

There can be different values of gasdynamic quantities either side of a shock wave. We wish to derive a relationship between these different values. To do this, we derive Rankine-Hugoniot jump conditions across a shock by looking at conservation laws.

Let P be some conserved quantity (such as mass or momentum) with flux Q, both functions of x and t. We can derive a conservation law by looking at balances in a region D in the (x,t) plane, whose boundary is C. This is done in [19], with the result

$$\int_{C} Pdx - Qdt = 0 \tag{2.37}$$

and the corresponding partial differential equation

$$\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} = 0. {(2.38)}$$

The latter is a conservation law.

Now rewrite (2.37) as

$$\frac{d}{dt} \int_{A}^{B} P dx = -[Q]_{A}^{B} \tag{2.39}$$

for points A and B being points representing the boundary of some fixed region V.

Consider a shock, S, in the (x,t) plane. If we take A=S and $B=S+\delta S$ as in Fig. 2.16, we obtain

$$[P^{-} - P^{+}]\delta S \approx -[Q]_{-}^{+}\delta t.$$
 (2.40)

Hence the shock speed will be given by

$$\frac{dS}{dt} = \frac{[Q]_{-}^{+}}{[P]_{-}^{+}}. (2.41)$$

²A classical solution of an nth order partial differential equation has to satisfy the partial differential equation and be n times continuously differentiable.

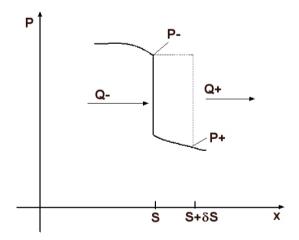


Figure 2.16: A shock.

Unfortunately, this approach does have its limitations. Suppose it was applied to the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (\frac{1}{2}u^2) = 0. \tag{2.42}$$

We would then get an expression for the shock speed as

$$U = \frac{\left[\frac{1}{2}u^2\right]_{-}^{+}}{\left[u\right]_{-}^{+}}.$$
 (2.43)

We can also rewrite (2.42) as

$$\frac{\partial}{\partial t}(\frac{1}{2}u^2) + \frac{\partial}{\partial x}(\frac{1}{3}u^3) = 0, \tag{2.44}$$

which would clearly lead to the shock speed being given by

$$U = \frac{\left[\frac{1}{3}u^{3}\right]_{-}^{+}}{\left[\frac{1}{2}u^{2}\right]_{-}^{+}}$$

$$\neq \frac{\left[\frac{1}{2}u^{2}\right]_{-}^{+}}{\left[u\right]_{-}^{+}}.$$
(2.45)

This shows that two conservation equations can be identical for smooth solutions though can have different solutions if we allow shocks!

A further property of these jump conditions is that we can have nonuniqueness of solutions for initial value problems. This can occur easily even for (2.42). Consider (2.42) with initial conditions

$$u(x,0) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} . \tag{2.46}$$

One possible solution is

$$u(x,t) = \begin{cases} 1 & x > t \\ \frac{x}{t} & t > x > 0. \\ 0 & 0 > x \end{cases}$$
 (2.47)

However, another solution is

$$u(x,t) = \begin{cases} 1 & x > \frac{t}{2} \\ 0 & x < \frac{t}{2} \end{cases}, \tag{2.48}$$

with the Rankine-Hugoniot condition

$$\frac{dx}{dt} = \frac{\left[\frac{1}{2}u^2\right]_{-}^{+}}{\left[u\right]_{-}^{+}} = \frac{1}{2},\tag{2.49}$$

hence we have nonuniqueness of the solution of the initial value problem.

2.2.3 Conservation laws for compressible flow

As was the case in the incompressible flow regime, we need to start with the conservation laws for compressible flow. We assume 1-dimensional flow, and obtain the conservation laws for mass, momentum and energy can be written as follows in conservation form (respectively) [19]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \tag{2.50}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(p + \rho u^2)}{\partial x} = 0 \tag{2.51}$$

$$\frac{\partial(\rho e + \frac{1}{2}\rho u^2)}{\partial t} + \frac{\partial(\rho u e + \frac{1}{2}\rho u^3 + pu)}{\partial r} = 0, \tag{2.52}$$

where e is the internal energy per unit mass of the gas.

Hence from (2.41) we can write the shock speed U as

$$U = \frac{[\rho u]_{-}^{+}}{[\rho]_{-}^{+}} \tag{2.53}$$

$$U = \frac{[p + \rho u^2]_{-}^{+}}{[\rho u]_{-}^{+}} \tag{2.54}$$

$$U = \frac{\left[\rho u e + \frac{1}{2}\rho u^3 + \rho u\right]_{-}^{+}}{\left[\rho e + \frac{1}{2}\rho u^2\right]_{-}^{+}}.$$
 (2.55)

Rearranging, and assuming $[u]_{-}^{+} \neq 0^{3}$, we arrive at the following Rankine-Hugoniot shock relations:

$$[\rho(U-u)]_{-}^{+} = 0 (2.56)$$

$$[p + \rho(U - u)^{2}]_{-}^{+} = 0 (2.57)$$

$$\left[h + \frac{1}{2}(U - u)^2\right]_{-}^{+} = 0 {(2.58)}$$

where $h = e + \frac{p}{\rho}$ is the enthalpy. Note that we don't need any area considerations as we had in (2.20) because shocks are so thin. Furthermore, these equations just express conservation of mass, momentum and energy across the shock (as the velocity term u - U is relative to the shock).

2.2.4 Thermodynamics background

Suppose we heat a gas. Denote the heat addition per unit mass by Q. The equation of conservation of energy (2.52), neglecting any conduction in the gas, generalises to [19]

$$\rho \frac{de}{dt} = \frac{p}{\rho} \frac{d\rho}{dt} + \rho \frac{dQ}{dt} \tag{2.59}$$

or

$$\frac{de}{dt} + p\frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{dQ}{dt} \tag{2.60}$$

where ρ is gas density, p pressure and u the velocity. For most perfect gases, it is experimentally confirmed that $e = c_v T$, where T is the temperature and c_v specific heat at constant volume, assumed constant.

For a perfect gas, we have the relationship

$$p = \rho RT \tag{2.61}$$

which is confirmed experimentally for a gas at rest. Here, R is the gas constant, equal to $c_p - c_v$ where c_p is the specific heat of constant pressure, assumed constant. Define $\gamma = \frac{c_p}{c_v} > 1$ ($\gamma = \frac{5}{3}$ for a monatomic gas). We will use this notation frequently later.

Using these relations, we can rewrite (2.60) in the more useful form

$$\frac{d}{dt}\log\left(\frac{p}{\rho^{\gamma}}\right) = \frac{(\gamma - 1)}{p}\rho\frac{dQ}{dt} = \frac{(\gamma - 1)}{R}\frac{1}{T}\frac{dQ}{dt}.$$
 (2.62)

We now define the *entropy* by

$$T\frac{dS}{dt} = \frac{dQ}{dt} \tag{2.63}$$

 $^{{}^{3}[}u]_{-}^{+}=0$ occurs in the case of a *contact discontinuity*, to be discussed later.

As we are working on an absolute temperature scale, T > 0. Also, as Q is the heat supplied volumetrically, we must have $\frac{dQ}{dt} > 0$. Hence $\frac{dS}{dt} > 0$, so entropy increases.

Combining (2.62) with this result and integrating, we obtain

$$\frac{p}{\rho^{\gamma}} = \exp\left(\frac{S - S_0}{c_v}\right) \tag{2.64}$$

where S_0 is a constant of integration.

Note that if there is no internal source term (Q = 0), then we have $\frac{dS}{dt} = 0$. This gives us the result that the entropy of a fluid particle is constant (isentropic flow).

We have discussed various conservation laws and their effect on jump conditions. We have not, however, considered what happens to the entropy across a shock. Using subscripts 1 for upstream, 2 for downstream, assuming the shock travels from region 2 to region 1, we can rearrange (2.56), (2.57) and (2.58) to get

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)M_1^2} \tag{2.65}$$

where $M_1 = \frac{U-u_1}{a_1}$ is the upstream Mach number and $a^2 = \frac{\gamma p}{\rho}$. Further, (2.57) and (2.58) together lead to

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}. (2.66)$$

Hence the entropy jump across a shock is obtained by considering

$$\left(\frac{p_2}{\rho_2^{\gamma}}\right) \left(\frac{\rho_1^{\gamma}}{p_1}\right) = \left(\frac{p_2}{p_1}\right) \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} \\
= \left(1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right) \left(\frac{1 + (\frac{\gamma - 1}{\gamma + 1})(M_1^2 - 1)}{1 + (M_1^2 - 1)}\right)^{\gamma}.$$
(2.67)

For values of the upstream Mach number M_1 near to 1, we may set $M_1^2 - 1 = \epsilon$ and expand the above expression asymptotically to get

$$\left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}\right)^{\gamma} = 1 + \frac{2\gamma(\gamma - 1)}{3(\gamma + 1)^2}\epsilon^3 + O(\epsilon^4)$$
 (2.68)

and hence the entropy jump is given as

$$\frac{S_2 - S_1}{c_v} = \log\left(\frac{p_2}{\rho_2^{\gamma}}\right) - \log\left(\frac{p_1}{\rho_1^{\gamma}}\right)$$

$$= O(\epsilon^3)$$

$$= O\left(\left(\frac{p_2 - p_1}{p_1}\right)^3\right) \tag{2.69}$$

by (2.66).

Interestingly, the result above gives us that we must have $M_1 > 1$ as the entropy must increase.

2.2.5 Simple wave flow

The final idea we will need is the concept of *simple wave flow*, which will be defined later in this section. The 1-dimensional conservation laws (2.50), (2.51) and (2.52) for $homentropic^4$ flow can be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 (2.70)$$

$$\rho(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}) = -\frac{\partial p}{\partial x}$$
 (2.71)

$$\frac{p}{\rho^{\gamma}} = \frac{p_0}{\rho_0^{\gamma}} = \text{constant},$$
 (2.72)

where p_0 and ρ_0 (assumed constant) are stagnation values of p and ρ .

Defining the speed of sound a for an isotropic flow by

$$a^2 = \frac{\gamma p}{\rho},\tag{2.73}$$

the conservation laws reduce to

$$\frac{\partial a}{\partial t} + u \frac{\partial a}{\partial x} + \frac{\gamma - 1}{2} a \frac{\partial u}{\partial x} = 0 {(2.74)}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{2a}{\gamma - 1} \frac{\partial a}{\partial x} = 0. {(2.75)}$$

We now have a system of partial differential equations in a and u. We can diagonalise the system using matrix algebra in the form

$$\left(\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right)(u \pm \frac{2a}{\gamma - 1}) = 0. \tag{2.76}$$

This shows that

$$u \pm \frac{2a}{\gamma - 1} = \text{constant on } \frac{dx}{dt} = u \pm a.$$
 (2.77)

 $u \pm \frac{2a}{\gamma - 1}$ are the Riemann invariants of the system, with characteristics $\frac{dx}{dt} = u \pm a$. We will refer to these characteristics as C_{\pm} . Depending on the boundary conditions and geometry, one family of characteristics can reduce to straight lines. This is *simple* wave flow.

When modelling shrapnel motion, we can consider the shrapnel as having no mass, but as a discontinuity between two regions. This motivates the next section on a shock tube, which will use ideas from simple wave flow and from section 2.2.2.

⁴A homentropic flow is one where the entropy S of all the fluid particles is constant for all time (cf. homo+entropy, Greek)

2.2.6 Shock tube

A shock tube is a tube consisting of, initially, two different regions, one of high pressure (compression chamber) and one of low pressure (expansion chamber). The regions are initially segregated by a diaphragm at x = 0, say (Fig. 2.17).

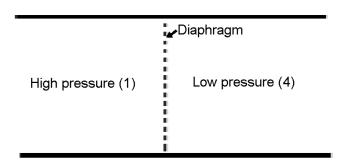


Figure 2.17: Shock tube before diaphragm bursts.

At time t = 0, the diaphragm is burst. The subsequent flow is divided into 4 different regions, labelled 1-4 as in Fig. 2.18. The interface between regions 2 and 3 is known as a *contact discontinuity* (or contact surface). It is the boundary between the fluids which were initially on opposite sides of the diaphragm. These fluids, neglecting any diffusion, will remain separated throughout the motion.

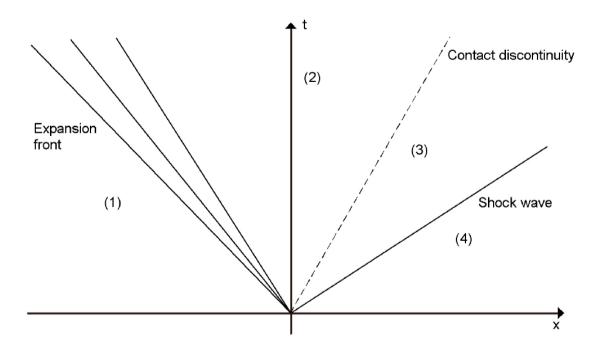


Figure 2.18: x - t diagram for a shock tube.

We have jump conditions across the discontinuity:

$$[p]_2^3 = [u]_2^3 = 0$$

$$[T]_2^3 \neq 0 \neq [\rho]_2^3.$$
 (2.78)

The boundary between regions 3 and 4 is a shock wave. We use Rankine-Hugoniot conditions (2.56), (2.57) and (2.58) across the shock (with U = 0) to give the following well-documented relations [14]

$$\frac{\rho_3}{\rho_4} = \frac{u_4}{u_3} = \frac{1 + \frac{\gamma_4 + 1}{\gamma_4 - 1} \frac{p_3}{p_4}}{\frac{\gamma_4 + 1}{\gamma_4 - 1} + \frac{p_3}{p_4}}$$
(2.79)

where subscripts denote the quantities in the different regions.

Further analysis which we omit here (though can be found in [14]) leads to the relationship

$$u_3 = a_4 \left(1 - \frac{p_3}{p_4}\right) \sqrt{\frac{\frac{2}{\gamma_4}}{(\gamma_4 + 1)\frac{p_3}{p_4} + \gamma_4 - 1}}.$$
 (2.80)

We now consider the boundary between regions (1) and (2), using our arguments on simple wave flow. By homentropy,

$$\rho = \rho_1 \left(\frac{a}{a_1}\right)^{\frac{2}{\gamma - 1}}.\tag{2.81}$$

The C_+ characteristics leave (1) and enter (2), so using (2.81), we arrive at

$$u_2 = \frac{2a_1}{\gamma_1 - 1} \left(\left(\frac{p_2}{p_1} \right)^{\frac{\gamma_1 - 1}{2\gamma_1}} - 1 \right). \tag{2.82}$$

Finally, imposing the conditions (2.78), we find a value for the velocity of the contact discontinuity.

2.2.7 Basic control volume equations

Ideally, this section would mirror the incompressible analysis and look at the cases of sudden expansions, contractions, inlets, orifices and screens as in sections 2.1.3, 2.1.4, 2.1.5 and 2.1.6. Literature does exist on pressure losses in screens and we will consider this in due course. Firstly, we need to consider what our jump conditions will be across a control volume.

For steady compressible flow, conservation of mass and momentum can be written

$$\nabla \cdot (\rho \mathbf{u}) = 0 \tag{2.83}$$

$$\rho(\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{u} = -\nabla p. \tag{2.84}$$

We consider a 2-dimensional control volume (Fig. 2.19), with upper and lower surfaces $y = h_{+}(x)$ and $y = h_{-}(x)$, respectively. The boundary condition, as before, will be

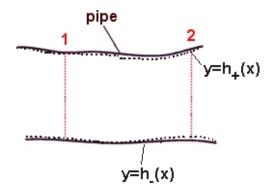


Figure 2.19: A control volume in a pipe.

 $\boldsymbol{u.n} = 0$ on $y = h_{\pm}$. Hence $v = uh'_{\pm}$ on $y = h_{\pm}$. Integrating (2.83) with respect to y between h_{-} and h_{+} and using the boundary condition gives

$$\frac{\partial}{\partial x}(\rho u(h_+ - h_-)) = 0. \tag{2.85}$$

Hence we obtain our first control volume equation, namely mass conservation in the form

$$A_1 \rho_1 u_1 = A_2 \rho_2 u_2, \tag{2.86}$$

where A_1 and A_2 represent the cross sectional areas at 1 and 2, respectively.

Denote the control volume by V, with boundary S. Integrating (2.84) over this volume and using (2.83) in a simple vector identity gives us

$$\int_{S} (p\mathbf{n} + \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n})) dS = 0.$$
 (2.87)

Hence, by using the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$, we recover

$$(p_1 + \rho_1 u_1^2) - (p_2 + \rho_2 u_2^2) = \text{Drag on pipe.}$$
 (2.88)

Our final equation will be a compressible form of Bernoulli's equation. For steady flow, we can integrate (2.84) along a streamline to obtain [19]

$$\frac{1}{2}u^2 + \int \frac{dp}{\rho} = \text{constant on a streamline.}$$
 (2.89)

The term $\int \frac{dp}{\rho}$ is the *enthalpy*. For homentropic flow, we have $p = k\rho^{\gamma}$ by (2.64) for some k. Hence, for such flow, we will have Bernoulli's equation in the form

$$\frac{1}{2}\boldsymbol{u}^2 + \frac{\gamma p}{(\gamma - 1)\rho} = \text{constant on a streamline.}$$
 (2.90)

Hence in our control volume, this says

$$\frac{\gamma p_1}{(\gamma - 1)\rho_1} + \frac{1}{2}u_1^2 = \frac{\gamma p_2}{(\gamma - 1)\rho_2} + \frac{1}{2}u_2^2. \tag{2.91}$$

Note that we cannot obtain the incompressible result by treating ρ as a constant, as this would give enthalpy h = h(p). This would contradict incompressible flow as in an incompressible flow, we can vary T and p independently. We can, however, retrieve the incompressible Bernoulli relation by taking

$$\lim_{\gamma \to \infty} \frac{\gamma p_1}{(\gamma - 1)\rho_1} = \frac{p_1}{\rho}.$$
 (2.92)

A more useful rearrangement of Bernoulli is

$$\frac{1}{2}u^2 + \frac{a^2}{\gamma - 1} = \frac{a_0^2}{\gamma - 1},\tag{2.93}$$

where a_0 is the stagnation speed of sound. This is speed of sound which would occur if the gas was brought to rest isentropically. Define the *Mach number* by

$$M = \frac{u}{a}. (2.94)$$

Hence by (2.93) we have

$$\frac{(\gamma - 1)}{2}M^2 + 1 = \frac{a_0^2}{a^2}. (2.95)$$

Equations for homentropic flow and the definition of a mean that we can write the flow variables p and ρ in terms of the Mach number, M, and their stagnation values, p_0 and ρ_0 , respectively, hence

$$p = p_0 \left(1 + \frac{(\gamma - 1)}{2} M^2\right)^{-\frac{\gamma}{\gamma - 1}} \tag{2.96}$$

$$\rho = \rho_0 \left(1 + \frac{(\gamma - 1)}{2} M^2\right)^{-\frac{1}{\gamma - 1}}.$$
 (2.97)

Recall in the incompressible case, we had an equation (2.25) expressing a head loss over certain pipe geometries. Clearly, the onset of turbulence in pipes in compressible flows will lead to some loss of pressure as we traverse downstream, depending on the pipe geometry for the same reasons as in incompressible flow. The incompressible flow expression for head loss was expressed in (2.25). The equivalent expression for compressible flow is

$$(h_1 + \frac{1}{2}u_1^2) - (h_2 + \frac{1}{2}u_2^2) = \tilde{h}_s$$
 (2.98)

for some compressible loss \tilde{h}_s , where h, the enthalpy, is defined by $h = \frac{\gamma p}{(\gamma - 1)\rho}$. As in the incompressible case, we postulate that we can write \tilde{h}_s in terms of the downstream velocity, say

$$\tilde{h}_s = \frac{1}{2}\tilde{K}u_2^2,\tag{2.99}$$

where \tilde{K} is a parameter determined experimentally. Clearly, as $\gamma \to \infty$, (2.98) will reduce to the incompressible expression (2.28) for some \tilde{K} .

At first sight, this may appear to contradict the first law of thermodynamics, which says that all of the energy changes in a system must add up to zero. We put forward the hypothesis that it does not, arguing that the internal dissipation of energy in the downstream turbulent flow balances the loss of enthalpy.

Obviously, if the flow remains laminar, we will have no losses and hence

$$\frac{\gamma p_1}{(\gamma - 1)\rho_1} + \frac{1}{2}u_1^2 = \frac{\gamma p_2}{(\gamma - 1)\rho_2} + \frac{1}{2}u_2^2. \tag{2.100}$$

We are now in a good position to discuss different pipe geometries, bearing in mind the different laws we have established. Conservation of mass (2.86) and momentum (2.88) will always hold, whereas the energy equation will depend on whether the flow is laminar (2.100) or whether it is turbulent (2.98).

2.2.8 Pipe flows

Consider a pipe (or long channel) with a slowly-varying cross-sectional area along its length. The cross-sectional area will be a function of position x, so conservation of mass (2.86) gives us

$$\rho A u = Q, \tag{2.101}$$

where Q is the mass flux (constant). We can substitute for p and ρ using (2.96) and (2.97), take logarithms and differentiate to obtain [19]

$$\frac{1}{A}\frac{dA}{dx} = \frac{(M^2 - 1)}{(1 + \frac{\gamma - 1}{2}M^2)M}\frac{dM}{dx}.$$
 (2.102)

This tells us that A can only have a minimum if M = 1 or $\frac{dM}{dx} = 0$. A plot of A against M is shown in Fig. 2.20. A takes its minimal value at A_c , where

$$A_c = \frac{Q}{\rho_0 a_0} \left(\frac{1+\gamma}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}.$$
 (2.103)

The graph also shows the crucial point that continuous are not possible for $A < A_c$. (2.102) tells us that for subsonic speeds (M < 1), a decrease in area (e.g. contraction in a pipe) will lead the Mach number M increasing (cf. fluid velocity increasing in incompressible flow) whereas for supersonic speed (M > 1), an increase in area will cause an increase in M. These ideas will be used in the next section on the Laval nozzle.

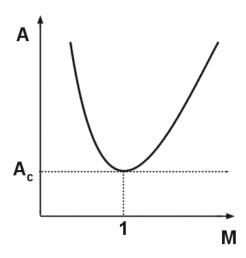


Figure 2.20: Plot of A against M.

(i) Laval nozzle

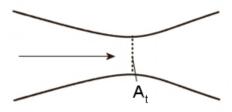


Figure 2.21: Laval Nozzle.

A Laval nozzle (Fig. 2.21) is a convergent-divergent pipe. The cross sectional area of the nozzle varies slowly along the length of the nozzle, which is connected

to a large reservoir of gas (which is at rest). The gas then flows through the nozzle. Turbulent flow in the expanding section of the pipe will lead to losses (2.98). Although we do not have any experimental data on \tilde{K} (2.99), we can still discuss possibilities for downstream flows which will depend strongly on the geometry of the pipe.

Continuous flows are determined by the difference in upstream and downstream pressure (recall (2.96)). We denote this difference by Δp . Let A_t denote the cross-sectional area of the throat, the part of the nozzle where cross-sectional area is a minimum. For continuous flow, we either have $A_t > A_c$ or $A_t = A_c$.

The first case causes the Mach number of the flow to increase to some maximum value as the pipe contracts, then decrease as the pipe expands. This maximum value, however, is always less than 1, so the flow remains subsonic. This is shown in Fig. 2.22.

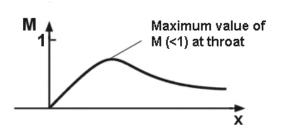


Figure 2.22: A schematic showing velocity when $A_t > A_c$ (so flow remains subsonic).

The interesting case of $A_t = A_c$ leads to the fluid velocity becoming sonic at the throat. Depending on the pressure, the flow out of the nozzle either decreases in velocity, or becomes supersonic (Fig. 2.23). The maximum fluid flux will occur at this value of A. Increasing Δp after this will not increase the flux through the nozzle.

Finally, if $A_t < A_c$, we cannot have a continuous flow, as shown in Fig. 2.20 and (2.102). Choking occurs in the nozzle as a consequence. This leads to shock waves being formed in the nozzle.

Fig. 2.24 shows a schematic of Δp against M_1 . For one fixed curve, say (i), we can see that as Δp increases, we approach the 'choking limit'. After this point, choking occurs and shocks are formed. The different curves (i), (ii) and (iii) show this phenomenon for different nozzle geometries. Curve (i) will be for a

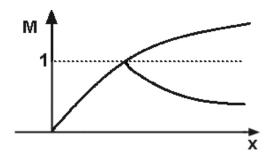


Figure 2.23: Area of throat $A_t = A_c$, hence sonic at the throat.

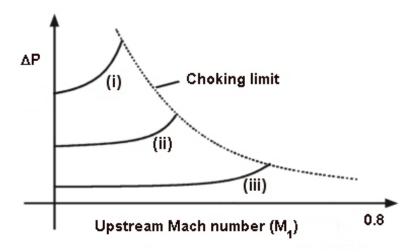


Figure 2.24: Schematic of Δp against M_1 , with choking limit shown.

nozzle with a higher degree of curvature than for (iii), which will have a more slowly-varying curvature.

Fig. 2.25 shows an outline of possible scenarios, based on increasing values of Δp .

In diagram (a), for the lowest Δp , $A_t > A_c$ and the flow is subsonic throughout the nozzle. Diagram (b) corresponds to the lower branch of Fig. 2.23. In (c), we have supersonic flow where the nozzle starts to expand. Further downstream, we will get a shock, which provides a boundary between M > 1 and M < 1. We can obviously apply our Rankine-Hugoniot jump conditions (2.56), (2.57) and (2.58) across this shock. Finally, for large values of Δp , we will get barrel shocks forming, so-called because of their shape. The geometry of these shocks is much

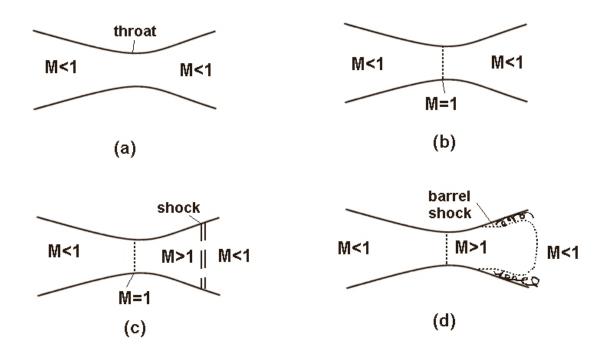


Figure 2.25: A sequence of different outcomes, each diagram representing an increasing value of Δp .

more complicated than in the normal shock in (c). Oblique shocks forming can lead to jet detachment [9]. This means that the outgoing flow has formed a jet, which is detached from the side walls of the nozzle. A shock reflected from the walls will form, making it nontrivial to solve.

The Laval nozzle is a common way to create supersonic flow, though much care is often taken in designing the diverging area so that the Mach number comes out as required, especially if we want to avoid oblique shock waves being formed downstream. Supersonic flow is normally avoided in industrial designs of piping systems, though taken advantage of in supersonic wind tunnel design.

We now try to apply the same ideas to the other flows we looked at in the incompressible case.

(ii) Gentle contraction

Clearly, if the incoming flow is subsonic, then the preceding analysis means that the flow will remain subsonic for a strict contracting pipe. We expect any losses (2.99) to be minimal, as in the incompressible case. Hence the flow will remain laminar and (2.100) will hold.

If the pipe contracts to a constant cross-sectional area, the flow can become

sonic here, admitting the possibility of shocks further downstream as in the case of the Laval nozzle. The nature of the downstream flow would depend on Δp and we expect similar results to those for the Laval nozzle.

(iii) Gentle expansion

For incoming subsonic flow, it can only become supersonic if we have a throat. This does not happen if the pipe is strictly expanding. Once again, we expect losses to be minimal, hence (2.100) will hold.

A diffuser, however, consists of straight pipe, followed by a slow expansion (Fig. 2.26). In this case, we can have choking, causing a shock further downstream. The effect of the turbulent flow downstream will obviously depend on the rate of increase of cross-sectional area of the diffuser.

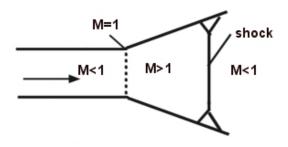


Figure 2.26: A diffuser.

(iv) Sudden contraction

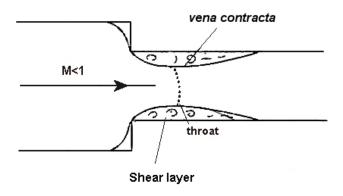


Figure 2.27: Sudden contraction with throat.

For a sudden contraction, we expect shocks to form in the downstream section of the pipe. Fig. 2.27 shows that there is, effectively, a throat. Hence, depending on Δp , we could have any of the outcomes we had in Fig. 2.25 for the Laval nozzle, such as barrel shocks or normal shocks forming downstream of the throat.

Turbulent flow will develop in the downstream section (cf. incompressible flow). Hence (2.99) will hold for some \tilde{K} , which will be determined experimentally.

(v) Sudden expansion

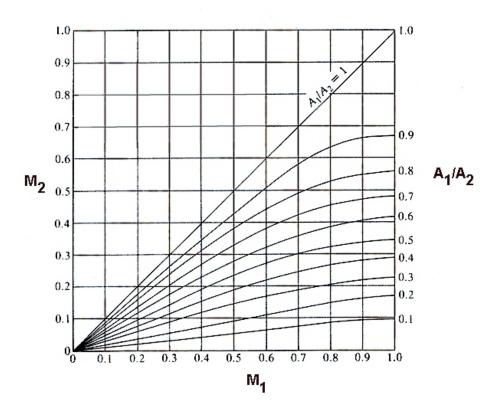


Figure 2.28: Sudden expansion (no choking) [16].

Denote upstream cross-sectional area of sudden contraction by A_1 , downstream cross-sectional area A_2 . Consider incoming subsonic flow. If there is no choking, we can obtain a plot of the upstream and downstream Mach numbers M_1 and M_2 [16], shown in Fig. 2.28.

Fig. 2.29 shows a possible set-up for a sudden expansion with choking. Incoming subsonic flow is incident on the sudden change in area. The downstream flow of this will, in all likelihood, be supersonic, although there are a few possibilities for the geometry of the shocks. Depending on Δp , we could get subsonic flow (relatively low Δp), normal shocks or barrel shocks. Fig. 2.30 shows

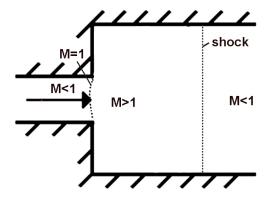


Figure 2.29: Sudden expansion with a possible downstream shock structure.

the possibility of oblique shocks, with a normal shock downstream. We know Rankine-Hugoniot relations across the normal shock ((2.56), (2.57) and (2.58)) and similar jump conditions will hold across oblique shocks [19]. Turbulent flow downstream will lead to pressure losses, (2.98), dependent on the ratio A_1/A_2 and on the upstream Mach number.

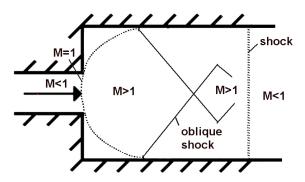


Figure 2.30: Oblique shocks.

(vi) Inlet

We expect inlets to have the same throat and shock geometry as for a sudden contraction. The nature and geometry of the shock, as well as depending on Δp (as for the Laval nozzle), will depend on the different types of entrance there are for an inlet (Fig. 2.11). For example, inlet (c) has a converging entrance. For incoming subsonic flow, this will lead to a sonic line at the throat which is where the inlet entrance stops contracting. Conversely, in (a), we expect the flow to change from subsonic to supersonic flow immediately with a shock wave

where the sudden change in area is. Similarly, the energy loss (2.98) will depend strongly on the geometry of the inlet entrance, as in section 2.1.3. Highly curved entrances to inlets will correspond to small values of \tilde{K} whereas sharp-edged entry points will mean larger values of \tilde{K} .

(vii) Orifices

Consider subsonic flow incident on an orifice. If choking occurs, Miller [16] notes that there will be a sonic line (where M=1) as in Fig. 2.31. We can expect oblique shocks and supersonic flow further downstream to this, turning subsonic over a normal shock even further downstream.

Turbulent motion behind the orifice plate (downstream) will occur, leading to energy losses in the form (2.98). Values of \tilde{K} will be experimentally determined.

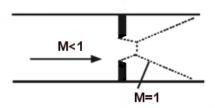


Figure 2.31: Sonic line (M = 1) shown for an orifice.

(viii) Empirical results for screens

Similar ideas on choking and losses (2.98) will apply for screens. We expect wakes to form behind screen elements, with jets between the wakes. As soon as the gas flows through the gaps between screen elements, choking can occur between each of the elements, leading to shock waves downstream. Again, the ideas discussed in nozzle flow and dependence on Δp will apply to a screen. The nature and evolution of the shocks will be very complicated as there are so many elements in the screen, resulting in interaction between the shocks.

The incoming upstream flow is laminar. Downstream, however, there is turbulent flow and hence (2.98) will hold. An alternative hypothesis on pressure losses is given by Cornell [8]. Define total pressure p_T by

$$p_T = p\left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma - 1}},$$
 (2.104)

which is p_0 in (2.96). Whereas we assumed p_0 was constant across the screen, Cornell postulates that it is not and hence there will be a total-pressure loss across a screen. He uses the nondimensional parameter

$$\lambda = \frac{\Delta p_T}{\frac{1}{2}\rho_1 u_1^2},\tag{2.105}$$

where Δp_T is the total pressure loss in a screen (which we can see is dependent on the upstream and downstream Mach numbers). Note that, in this case, λ is based on the upstream velocity u_1 .

We can rewrite (2.98) in terms of the total pressure as

$$\left(\frac{\gamma}{\gamma - 1} \left(\frac{p_1}{\rho_1^{\gamma}}\right)^{\frac{1}{\gamma}} p_{T_1}^{\frac{\gamma - 1}{\gamma}}\right) - \left(\frac{\gamma}{\gamma - 1} \left(\frac{p_2}{\rho_2^{\gamma}}\right)^{\frac{1}{\gamma}} p_{T_2}^{\frac{\gamma - 1}{\gamma}}\right) = \frac{1}{2} \tilde{K} u_2^2,$$
(2.106)

where p_{T_1} and p_{T_2} are the upstream and downstream total pressures, respectively. This is similar to (2.105), which can be written as

$$p_{T_1} - p_{T_2} = \frac{1}{2} \lambda \rho_1 u_1^2. \tag{2.107}$$

The difference is that Cornell's hypothesis is that there will be a loss in total pressure (head) whereas our hypothesis predicts a loss in the energy over the screen.

For low Mach-number flow past a round-wire screen, Cornell looks at a method developed by Wieghardt. Data correlated for 0.006 < Re < 1000 leads to the relation

$$\lambda = \frac{33.93}{Re} \frac{S(1-S)^{-1.27}}{1+\sqrt{(1-S)}},\tag{2.108}$$

where S is the screen solidity (2.35). Graphically, these relationships are seen to be reasonable correlations.

For high incident Mach number (though still subsonic) flow through a roundwire screen, the effect of the compressibility plays more of a role. As we would expect, it is found that, for a given S, the loss coefficient λ increases with the incident Mach number M_1 . Furthermore, the compressibility is more prevalent at high values of S, as this causes very high velocities in the screen passages.

Fig. 2.32 shows experimental results Cornell cites from Adler which are based on high-velocity air-flow tests. The dotted line shown indicates the *choke limit*, which shows where choked flow occurs (*cf.* nozzle). This graph corresponds with the schematic (Fig. 2.24) which showed the limit on choked flow as a function of Δp .

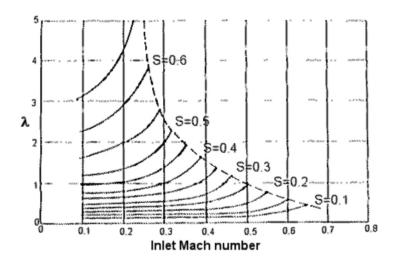


Figure 2.32: Round-wire screen losses in high-velocity flow [8] (experimental).

Chapter 3

Models for shrapnel

We have described some background material which may be useful for formulating a theory on shrapnel motion. We now present some possible models, starting by modelling the shrapnel as a piston, both in the incompressible and compressible case. The models will be 1-dimensional, following on from the analysis in chapter 2. The first set of models consider a relatively thin casing which will fragment into a thin layer of shrapnel. An alternative approach of looking at ideas used in rock blasting will follow, before concluding the chapter with a comparison of the various models.

3.1 Expanding gas causing piston motion into a vacuum

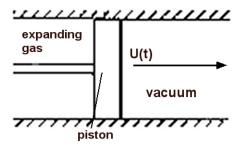


Figure 3.1: Piston pushed by expanding gas into a vacuum.

The simplest paradigm is to model the shrapnel as a piston of mass m (per unit area), position X(t), velocity U(t) being pushed by the initially uniform high pressure gas into a vacuum to the right of the piston. We can apply the theory on simple wave flow (section 2.2.5) because this gas flow is homentropic (that is, entropy (2.64) of the fluid particles is constant for all time).

So we have

$$\frac{p}{\rho^{\gamma}} = \frac{p_0}{\rho_0^{\gamma}} \tag{3.1}$$

where p_0 and ρ_0 are the rest values of p and ρ , respectively, in the undisturbed gas to the left of the piston.

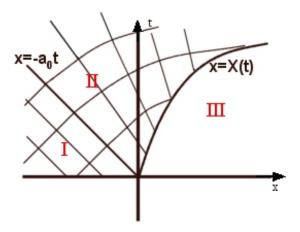


Figure 3.2: Characteristic Diagram.

The characteristic diagram will be divided into three regions, as shown in Fig. 3.2. Region I, undisturbed, where $x < -a_0t$, region II where $-a_0t < x < X(t)$ and region III where X(t) < x.

In region I, the information from the initially stationary piston and undisturbed gas gives us u = 0 and $a = a_0$. Hence in this region, we have

$$u - \frac{2a}{\gamma - 1} = -\frac{2a_0}{\gamma - 1} \text{ on } C_-$$
 (3.2)

$$u + \frac{2a}{\gamma - 1} = \frac{2a_0}{\gamma - 1} \text{ on } C_+.$$
 (3.3)

The positive characteristics C_+ leave region I and go into region II. Hence by looking at the C_+ expression (3.3) in region I, we obtain

$$a = a_0 - \frac{1}{2}(\gamma - 1)u \tag{3.4}$$

which must also hold in region II.

Hence, proceeding as in simple wave flow, we can get expressions for ρ and p in terms of a, a_0 and ρ_0 to arrive at

$$\rho = \rho_0 \left(1 - \frac{1}{2} (\gamma - 1) \frac{u}{a_0} \right)^{\frac{2}{\gamma - 1}} \tag{3.5}$$

$$p = p_0 \left(1 - \frac{1}{2} (\gamma - 1) \frac{u}{a_0} \right)^{\frac{2\gamma}{\gamma - 1}}. \tag{3.6}$$

The boundary conditions are that the force on the piston is the product of mass and acceleration (Newton), that the gas velocity on the piston is equal to the piston velocity and that the initial piston velocity is zero. These can be written as

$$m\dot{U} = p \tag{3.7}$$

$$u = U$$
 on $x = X(t)$ (3.8)

$$U(t) = 0$$
 at $t = 0$. (3.9)

Hence

$$m\frac{dU(t)}{dt} = p_0 \left(1 - \frac{1}{2}(\gamma - 1)\frac{U(t)}{a_0}\right)^{\frac{2\gamma}{\gamma - 1}}.$$
 (3.10)

This can be integrated and using the initial condition we obtain

$$U(t) = \frac{2a_0}{\gamma - 1} \left(1 - \left(1 + \left(\frac{\gamma + 1}{2ma_0} \right) p_0 t \right)^{-\frac{\gamma - 1}{\gamma + 1}} \right). \tag{3.11}$$

Nondimensionalise the solution by scaling U with a_0 and t with $\frac{ma_0}{p_0}$, giving

$$\tilde{U}(t) = \frac{2}{\gamma - 1} \left(1 - \left(1 + \left(\frac{\gamma + 1}{2} \right) \tilde{t} \right)^{-\frac{\gamma - 1}{\gamma + 1}} \right). \tag{3.12}$$

A plot of the nondimensional solution is shown in Fig. 3.3. The shape of the graph shows us that the piston's velocity initially increases very quickly and then tends to a limit. It is trivial to show that (in the dimensional case)

$$U(t) \to \frac{2a_0}{\gamma - 1} \text{ as } t \to \infty.$$
 (3.13)

Let M_p be the Mach number of the piston, $M_p = \frac{U}{a}$. So using (3.6) and (3.4) we can see that

$$\frac{2a_0}{(\gamma - 1)a} = \frac{2(a + \frac{1}{2}(\gamma - 1)u)}{(\gamma - 1)a}
= \frac{2}{\gamma - 1} \left(\frac{p_0}{p}\right)^{\frac{\gamma - 1}{2\gamma}}.$$
(3.14)

Hence (3.11) gives

$$M_p = \frac{2}{\gamma - 1} \left(\frac{p_0}{p}\right)^{\frac{\gamma - 1}{2\gamma}} \left(1 - \left(1 + \left(\frac{\gamma + 1}{2ma_0}\right)p_0t\right)^{-\frac{\gamma - 1}{\gamma + 1}}\right). \tag{3.15}$$

Using (3.6) and (3.11) we see that the pressure on the piston is given by

$$p(t) = p_0 \left(1 - \frac{1}{2} (\gamma - 1) \frac{U(t)}{a_0} \right)^{\frac{2\gamma}{\gamma - 1}}$$

$$= p_0 \left(1 + \frac{\gamma + 1}{2ma_0} p_0 t \right)^{-\frac{2\gamma}{\gamma + 1}}.$$
(3.16)

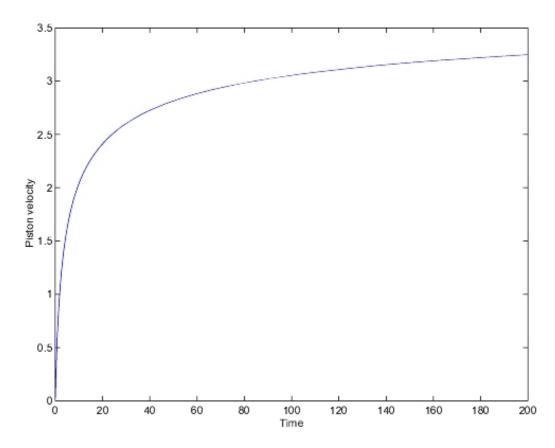


Figure 3.3: A plot showing (nondimensional) piston velocity as a function of (nondimensional) time.

Hence we know the ratio $\frac{p_0}{p}$, so substituting into (3.15) we observe that

$$M_p = \frac{2}{\gamma - 1} \left(\left(1 + \frac{\gamma + 1}{2ma_0} p_0 t \right)^{\frac{\gamma - 1}{\gamma + 1}} - 1 \right). \tag{3.17}$$

Expanding (3.17) we find that

$$\frac{p_0 t}{m a_0} \ll 1 \quad \Rightarrow \quad M_p < 1 \tag{3.18}$$

$$\frac{p_0 t}{m a_0} \gg 1 \quad \Rightarrow \quad M_p > 1. \tag{3.19}$$

$$\frac{p_0 t}{m a_0} \gg 1 \quad \Rightarrow \quad M_p > 1. \tag{3.19}$$

So the piston can be either subsonic or supersonic. Note that for small t, we will always have $M_p < 1$ (so always starts subsonic) and for sufficiently large $t, M_p > 1$ (supersonic).

The model we have just discussed considers the shrapnel, of mass m, as being a solid boundary between a region of high pressure and a vacuum. Another approach would be to model the shrapnel as a discontinuity (of no mass) between two regions of different gas densities. This is the shock tube (section 2.2.6). As shown earlier, this predicts a constant value for the shrapnel velocity.

We could try to model a piston in a shock tube (section 2.2.6). By this, we mean replacing the contact discontinuity in the shock tube (Fig. 2.18) with a piston of mass m. At first glance, this would seem to be a good model to try, and we would use $p_2 - p_3 = m\dot{U}$ as the condition on the piston. The accelerating piston leads to a shock wave (as before) and an expansion wave. We will not, however, have the luxury of homentropic flow everywhere (the piston will be accelerating) and hence the problem would have to be solved computationally.

The ideas of the piston moving into a vacuum and the shock tube (with contact discontinuity), in tandem, should give lower and upper bounds for the shrapnel velocity. What we would like, however, is a model which gives an intermediate value, corresponding with gas flowing between metal fragments. This motivates the idea of a porous piston which is, in effect, a screen.

3.2 Porous Piston

We consider an improvement to the model by looking at a porous piston of mass m. This is an improvement to the previous models as there will be many fragments (shrapnel), which we can model as the porous piston. In effect, our porous piston is a moveable screen. Once again, suppose the piston has position x = X(t), velocity U(t). Firstly, for simplicity and to gain intuition, we look at incompressible gas flow.

3.2.1 Incompressible flow and the porous piston

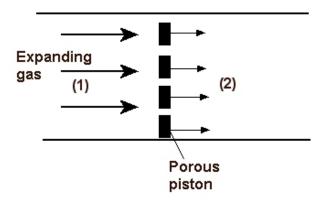


Figure 3.4: A porous piston (screen).

We model the porous piston as a discontinuity, and so proceeding as in chapter 2, we would expect to have three jump conditions across the piston. Mass conservation and the head loss equation (relative to the piston) (2.28) give

$$\rho(u_1 - U) - \rho(u_2 - U) = 0 (3.20)$$

$$\left(\frac{p_1}{\rho} + \frac{(u_1 - U)^2}{2}\right) - \left(\frac{p_2}{\rho} + \frac{(u_2 - U)^2}{2}\right) = K\frac{(u_2 - U)^2}{2}.$$
 (3.21)

By balancing momentum, we have

$$(p_1 + \rho(u_1 - U)^2) - (p_2 + \rho(u_2 - U)^2) = m\dot{U} + D, \tag{3.22}$$

where D is the drag on the piston and on the walls. We have neglected the acceleration of the gas through the porous piston, yet included the acceleration of the piston itself. We can justify this as the acceleration of the gas is negligible compared to the acceleration of the piston. We now solve for U with the boundary condition

$$U(0) = 0. (3.23)$$

The first of the equations gives us that $u_1 = u_2$, so rearranging the second two equations we get

$$m\dot{U} = \frac{1}{2}\rho K(u_1 - U)^2 - D. \tag{3.24}$$

We consider two cases. Firstly, if the drag D is zero, we can integrate (3.24) with the boundary condition to get a solution

$$U = u_1 - \frac{1}{\frac{1}{u_1} + \frac{\rho K}{2m}t}. (3.25)$$

Note firstly that U is bounded above, and that as $t \to \infty$, $U \to u_1$, so, not surprisingly, the piston speed cannot exceed the gas speed (Fig. 3.5).

Now solve with the drag term D nonzero. One hypothesis of flow through a screen [5] is to write the drag term in terms of the square of the gas velocity (relative to the piston), hence

$$D = \frac{1}{2}\rho C_d (u_2 - U)^2. \tag{3.26}$$

This leads to a solution for U in the form

$$U = u_1 - \frac{1}{\frac{1}{u_1} + \frac{\rho(K - C_d)}{2m}t}.$$
(3.27)

Once again, as $t \to \infty$, $U \to u_1$. This solution is physically invalid if $K < C_d$, as otherwise U is negative. Whether the condition $K < C_d$ can physically happen or not will be resolved experimentally by calculating values for C_d and K. We note that both constants will depend on the geometry of the 'porous piston' (the solidity in particular).

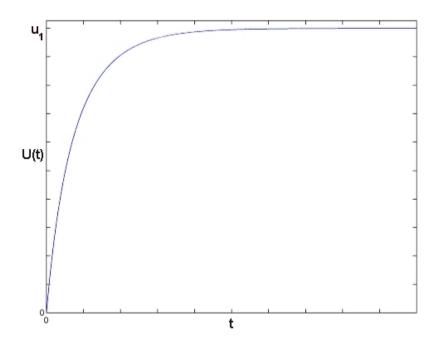


Figure 3.5: U(t) for incompressible flow.

3.2.2 Compressible flow and the porous piston

We now consider the more realistic case of compressible flow and the porous piston. Looking at a screen which admits compressible flow is a lot trickier than the incompressible case, and unfortunately the complete analysis must be left as an open question. We can, however, continue with the ideas we have used up until now to gain some useful insights on shrapnel motion.

Fig. 3.4, as well as being the basic configuration for the incompressible case, also shows the set-up for the compressible case locally around the piston. We discuss this area first before going on to the downstream flow.

The jump conditions on the piston are, in general, difficult to work out. We will obviously have conservation of mass relative to the screen, hence we modify (2.86) as

$$\rho_1(u_1 - U) - \rho_2(u_2 - U) = 0. (3.28)$$

Similarly, using the hypotheses of chapter 2, (2.98) becomes

$$\left(\frac{\gamma p_1}{(\gamma - 1)\rho_1} + \frac{1}{2}(u_1 - U)^2\right) - \left(\frac{\gamma p_2}{(\gamma - 1)\rho_2} + \frac{1}{2}(u_2 - U)^2\right) = \tilde{h}_s,\tag{3.29}$$

where \tilde{h}_s is as in (2.99).

The problematic condition is the momentum condition. Recall the force balance for the piston pushed by gas expansion into a vacuum $p_1 - p_2 = m\dot{U}$ and across a

shock was $[p + \rho(u - U)^2]_1^2 = 0$. In light of this, we might try a relationship of the form

$$(p_1 + \mu \rho (u_1 - U)^2) - (p_2 + \mu \rho (u_2 - U)^2) = m\dot{U}$$
(3.30)

for a constant μ satisfying $0 \le \mu \le 1$. Here, μ would be a measure of how porous the piston is. We see that $\mu = 0$ would correspond to the piston expanding into a vacuum, whereas $\mu = 1$ would be equivalent to a shock wave (with a drag term). Unfortunately, the use of such a factor cannot be justified by momentum balances. Hence, we use the momentum balance given by

$$(p_1 + \rho_1(u_1 - U)^2) - (p_2 + \rho_2(u_2 - U)^2) = D + m\dot{U}.$$
(3.31)

We use the hypothesis [5] that the drag term will be written in terms of the square of the downstream velocity and the downstream density for some drag coefficient C_d , so

$$D = \frac{1}{2}\rho_2 C_d (u_2 - U)^2. \tag{3.32}$$

Thus we have our jump conditions across the piston.

Before looking at the downstream motion, it is of importance to look at the entropy jump across the piston. In the case of a shock wave, we have already obtained an expression for the entropy jump across the shock (2.69). We need to reinvestigate this now we have the extra terms D, $m\dot{U}$ and \tilde{h}_s in the jump conditions across the piston and wish to find out their effect.

Using the notation $\epsilon = M_1^2 - 1$ (M_1 is the Mach number relative to the screen) and introducing the nondimensional parameter

$$\xi = \left(\frac{m\dot{U} + D - (\gamma - 1)\tilde{h}_s \rho_2}{p_1}\right),\tag{3.33}$$

we can mirror the analysis from before and manipulate Rankine-Hugoniot relations to obtain the ratios $\frac{p_2}{p_1}$ and $\frac{\rho_1}{\rho_2}$. For small ξ , rearrangement of (3.28), (3.29) and (3.31) give lead to

$$\left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}\right)^{\gamma} \sim 1 + O(\xi). \tag{3.34}$$

Hence the entropy jump will be $O(\xi)$ for ξ sufficiently small. Obviously, if $\xi \sim \epsilon^3$ (or smaller), then the entropy jump is what it was with no drag (D), loss (\tilde{h}_s) or acceleration term $(m\dot{U})$, namely $O(\epsilon^3)$.

For larger values of ξ , we can see that ξ will dominate the entropy jump, which will be an important consideration in modelling the gas flow behind the screen. This now leads us on to discussing possible downstream flow regimes.

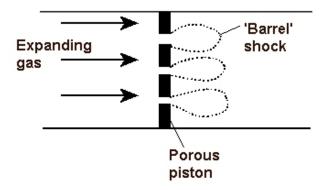


Figure 3.6: A porous piston (screen) with barrel shocks.

Fig. 3.6 shows what we expect will happen after a short time. Choking can occur in the screen, causing the flow to go from subsonic to supersonic. We then suppose that barrel shocks (as in section 2.2.8) will form. These are shock waves formed locally around the exit points of the screen. The geometry of these barrel shocks is not known exactly, presenting us immediately with huge difficulties. As well as having jump conditions across the porous piston, we will have jump conditions across the barrel shocks. These are nontrivial.

The large time behaviour of the shocks is likely to be very different. We expect the barrels to merge, leading to an arrangement as in Fig. 3.7. The flow here is divided

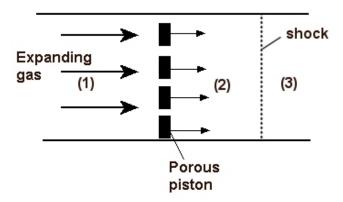


Figure 3.7: A porous piston (screen) with a shock to the right.

into three distinct regions. The first two regions are as discussed above, where the gas rushes through the piston into region 2 and causes the piston to move with high velocity. There will be a shock further downstream, which divides regions 2 and 3.

Rankine-Hugoniot shock conditions ((2.56), (2.57) and (2.58)) can be applied at the boundary between 2 and 3, giving us three relationships between our gasdynamic variables in the usual manner. To attempt to solve the system, we would try to solve (3.31) for the piston with the jump conditions and shock conditions. We know that there is a jump in entropy across the porous piston. This means that we cannot use the Riemann invariants from (1) in region (2), as we unfortunately do not have homentropic flow. What we do have is unknowns and equations relating them. We cannot, however, solve the equations in region (2) analytically. Essentially, all we have is boundary conditions at both boundaries of region (2) (from the shock jump conditions and from the piston jump conditions) to accompany the general equations of gasdynamics which hold in (2). These would have to be solved computationally.

In summary, we have developed a model for compressible gas flow through a moveable screen. We have jump conditions (3.28), (3.29) and (3.31) which relate pressures, densities and gas velocities either side of the piston. The drag coefficient C_d and \tilde{K} need to be found experimentally. Downstream, we have shock waves which, in the simplest case, will be as in Fig. 3.7. For this case, we can solve computationally using Rankine-Hugoniot conditions across the shock (2.56), (2.57) and (2.58). Realistically, we will have a more complicated shock-structure downstream.

3.3 Rock blasting theory

Our previous models consider gas flowing through thin layers of shrapnel, and hence would be best applied to thin casings. The following model is on cliff blasting. It will become clear that the theory developed applies more to thicker casings.

The processes involved in rock blasting are important in the mining industry. This has motivated applied mathematicians to look at the problem and try and predict the location of the rocks after blasting so that machinery can extract the ore with the utmost efficiency. The mathematics involved is different to the theory we have been looking at so far, though we can hopefully justify applying it to the shrapnel problem. Here we look at an existing paper [20] which looks at the processes involved when blasting a cliff.

3.3.1 Cliff Blasting

Before blasting takes place, the cliff must be prepared by drilling an array of vertical boreholes in the cliff. These boreholes will normally stretch at least half of the cliff height. The explosive charges are placed in the boreholes, which are then covered with loose rocks. We are now ready to detonate the explosive in some prearranged sequence.

The first thing that happens after the charge burns rapidly is that elastic waves will propagate from the borehole into cliff, causing fracturing to take place. Fig. 3.8 shows this process, although in practice the wave will reflect off fractures and internal cliff inhomogeneities and hence will not necessarily be strictly vertical as shown. Also, depending on the borehole, we will get propagation into terra firma (the solid rock which won't be cracked), though reflections from this will be neglected in the following model. This stage lasts about 10 milliseconds, and, depending on the size of the cliff, all an observer would see from this stage would be the cliff vibrate.

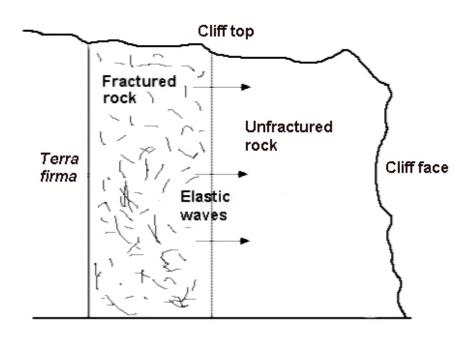


Figure 3.8: Elastic/plastic wave propagation leading to fracturing.

The next thing an observer would see is puffs of smoke (or dust) emerging from the external boundaries of the cliff. The high pressure gas caused by the burning of the charge has propagated through the cliff to cause this. In doing so, the gas links up some of the existing fractures and may create additional fractures. We might expect the fully fractured rocks to start accelerating at this stage, but this does not happen as they are held securely in place by the rock which the gas has not reached yet. This stage will last in the order of 100 milliseconds. The rock is now ready to move. This will occur when the cliff cannot hold the high pressure gas. The build up of pressure in the cliff is relieved as the completely fractured rock is rapidly accelerated and will begin to move, as shown in Fig. 3.9. The red line shows the movement of the front where rock acceleration starts to take place. This rock acceleration will take in the order of 10 milliseconds.

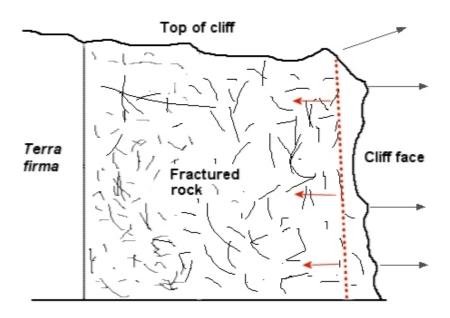


Figure 3.9: Rocks from the fully fractured cliff start to move, starting from the right-hand face.

Finally, the rocks which are now in motion will free-fall under gravity. They will form a "muck-pile", allowing valuable ore to be obtained with relative ease. The free-fall section will last the order of one second.

3.3.2 Existing model

An existing model [20] analyses the high pressure gas propagation and the build up of pressure leading to rock movement in a cliff. The model uses the idea of two-phase flow, which, as the name suggests, refers to there being two phases of a substance in a flow. Two-phase flow typically involves the phases being liquid and gas, though in the cliff blasting case, the different phases are the gas and the rock. Often, the flow is turbulent which leads to some kind of averaging being essential to model the mean flow [11]. We do not average as such here, though need to introduce a new variable α , the *void fraction* which is the gas volume fraction.

To set up the two-phase flow model, we first need to make some assumptions about the cliff and gas. Firstly, the cliff is assumed to be homogeneous. We also need to assume that the rocks aren't porous, and that the gas flow is limited to the cracks between the rocks only. This assumption is quite a large one, as many rocks tend to be very porous, which would have consequences on the gas propagation. It is also assumed that no cracking occurs in this stage and that the distribution of the cracks is known from the earlier stage.

Hence the basic equations of motion come from conservation of mass and momentum for the rock and for the gas. Mass conservation gives

$$\frac{D_g}{Dt}(\alpha \rho_g) + \alpha \rho_g \nabla \cdot \boldsymbol{v} = 0 \tag{3.35}$$

$$\frac{D_r}{Dt}((1-\alpha)\rho_r) + (1-\alpha)\rho_r \nabla \cdot \boldsymbol{u} = 0$$
(3.36)

where subscripts g and r refer to the gas and rock respectively, \mathbf{v} and \mathbf{u} are the gas and rock velocities (respectively), α is the volume fraction of gas and $\frac{D}{Dt}$ denotes the material derivative for the respective phase.

Neglecting any drag between the phases and assuming that the pressure is the same in both phases, we can similarly use conservation of momentum to obtain

$$\rho_g \alpha \frac{D \boldsymbol{v}}{D t} + \alpha \boldsymbol{\nabla} p = 0 \tag{3.37}$$

$$\rho_r(1-\alpha)\frac{D\boldsymbol{u}}{Dt} + (1-\alpha)\boldsymbol{\nabla}p = 0. \tag{3.38}$$

An important nondimensional parameter is given by

$$\beta = \frac{Kp_0\rho_{r_0}}{\rho_0} \tag{3.39}$$

where K is proportional to the Young's Modulus of the rock, ρ_{r_0} the initial rock density, ρ_0 the initial gas density in the borehole and p_0 the initial pressure in the borehole. The Young's modulus is a measure of the stiffness of the material (rock in our case) as a function of its density. Kp_0 will be used as a suitable scale for the void fraction. Typically, for rocks, β has a value of the order 0.01 and the Young's modulus for rocks is of the order $10^7 Pa$.

Assuming the gas is ideal, we will have $\frac{p}{p_0} = (\frac{\rho}{\rho_0})^{\gamma}$. Nondimensionalisation of the equations will give us a hyperbolic system for the motion of the gas and rock. In choosing appropriate scales for nondimensionalisation, we will have different choices

to make depending on whether we are dealing with the stage where the rocks are free to move or not.

The more interesting stage for us will involve the rocks being free to move. Looking at one-dimensional equations we arrive at, taking the limit as $\beta \to 0$,

$$\frac{\partial \rho}{\partial t} + a(\rho) \frac{\partial u}{\partial x} = 0 \tag{3.40}$$

$$\frac{\partial u}{\partial t} + b(\rho) \frac{\partial \rho}{\partial x} = 0 ag{3.41}$$

where $a(\rho) = (\gamma \rho^{\gamma-1} + \frac{1}{\rho^2})^{-1}$ and $b(\rho) = \gamma \rho^{\gamma-1}$.

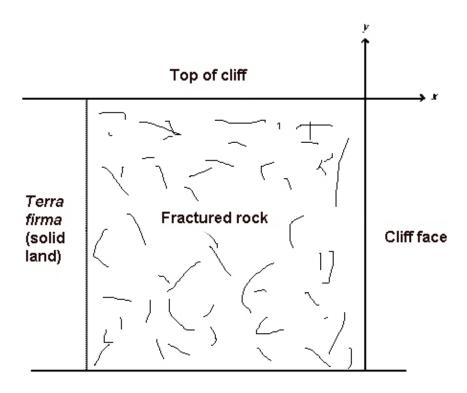


Figure 3.10: Cliff considered as a 2D model.

To solve this hyperbolic system, we need to impose some initial conditions. We make the following simplifications to the problem in order to enable us so obtain an analytic solution. Firstly, we assume that the cliff area is square, and that two sides of the square will be external surfaces (Fig. 3.10). We also assume that gas pressure and void fraction are initially spatially uniform, so we have the boundary condition

 $\alpha(x,0) = 1 = \rho(x,0)$. For simplicity, the inhomogeneity owing to the borehole is neglected.

The boundary condition used is that the pressure on the external surfaces of the cliff is atmospheric pressure (p_{atmos}) , say

$$p = p_{atmos}/p_0 = \epsilon^{\gamma} \tag{3.42}$$

and that the boundaries by terra firma have $\mathbf{u} \cdot \mathbf{n} = 0$ (noting that p is nondimensional in (3.42)).

Substituting $\rho = g(s(x,t))$ and choosing $\frac{dg}{ds} = \left(\frac{a(g)}{b(g)}\right)^{\frac{1}{2}}$ we can obtain the Riemann invariants of the system via

$$\left(\frac{\partial}{\partial t} \pm (ab)^{\frac{1}{2}} \frac{\partial}{\partial x}\right) (u \pm s) = 0. \tag{3.43}$$

Hence we have simple wave flow (section 2.2.5) as $u \pm s = \text{constant on } \frac{dx}{dt} = \pm (ab)^{\frac{1}{2}}$ and so using the initial conditions on α and ρ , we obtain the a characteristic diagram (Fig. 3.11) which leads to an analytical solution:

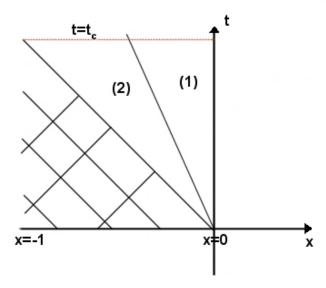


Figure 3.11: Characteristic diagram for cliff fracturing.

$$\rho = \begin{cases}
\epsilon & -\left(\frac{\gamma\epsilon^{\gamma+1}}{\gamma\epsilon^{\gamma+1}+1}\right)^{\frac{1}{2}} < \frac{x}{t} < 0 \\
\left(\frac{\left(\frac{x}{t}\right)^{2}}{\gamma(1-\left(\frac{x}{t}\right)^{2})}\right)^{\frac{1}{\gamma+1}} & -\left(\frac{\gamma}{\gamma+1}\right)^{\frac{1}{2}} < \frac{x}{t} < -\left(\frac{\gamma\epsilon^{\gamma+1}}{\gamma\epsilon^{\gamma+1}+1}\right)^{\frac{1}{2}}
\end{cases} (2)$$

$$u = \int_{\rho}^{1} z^{\frac{\gamma - 3}{2}} (\gamma^{2} z^{\gamma + 1} + \gamma)^{\frac{1}{2}} dz$$
 (3.45)

Reflections off the face x = -1 have been neglected. Hence this solution is only valid until information from the face at x = 0 has reached x = -1, at time $t = t_c$.

For $\gamma = 1$, we can calculate the integral and get that $u \sim \log \epsilon + O(1)$ as $\epsilon \to 0$ We can also calculate the velocity at the edge of the cliff by substituting $\rho = \epsilon$ into (3.45).

Looking at plots (Fig. 3.12) of the void fraction and rock velocity for different times, we observe that the rocks on the outermost layer of the cliff fly off with a very high velocity (almost like a boundary layer), whereas those nearer to x = -1 won't fly off until a later time. This is what we expect physically, and indeed it is what is seen.

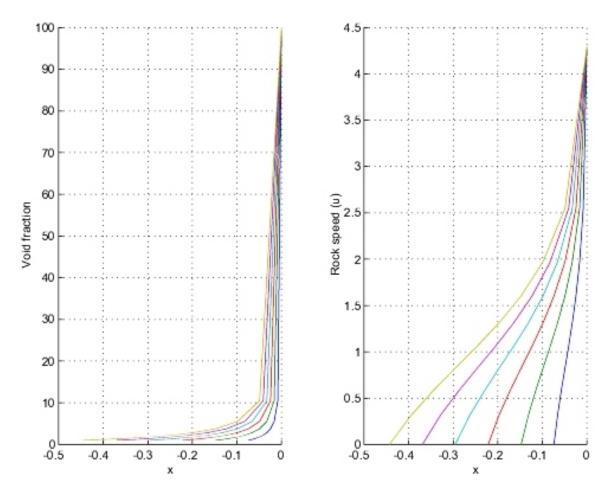


Figure 3.12: Graphs showing void fraction (scaled) and rock speed (scaled), as functions of position for different values of time. The blue line indicates the smallest time value, then working up to the olive line. γ is taken to be 1.3.

3.3.3 Rock blasting theory applied to shrapnel

As mentioned earlier, the main new feature of this blasting model is that we are no longer considering gas rushing past a thin layer of shrapnel. We now have a two-phase model which essentially considers the cliff as a big block containing many fragments. As a consequence, this theory lends itself to being applied to thicker casings.

The idea of a borehole at one end of the cliff suggests that the theory would best apply to the shrapnel problem with a cylindrical casing detonated at one end. However, the assumptions made in the one dimensional cliff model (neglecting the borehole and assuming homogeneity) mean that the ideas will follow in a spherical bomb too.

Recall that in the rock blasting model, we had a borehole at one end of the cliff. It is trivial to adapt this analysis to apply so that the initial wave from the borehole propagates (still 1D) in both directions. If we take the left hand face at x = -R, say, and applying the boundary condition (3.42) at this new face, a solution is obtained

$$\rho = \begin{cases}
\epsilon & -\left(\frac{\gamma\epsilon^{\gamma+1}}{\gamma\epsilon^{\gamma+1}+1}\right)^{\frac{1}{2}} < \frac{x}{t} < 0, 0 < \left(\frac{x+R}{t}\right) < \left(\frac{\gamma\epsilon^{\gamma+1}}{\gamma\epsilon^{\gamma+1}+1}\right)^{\frac{1}{2}} \\
\left(\frac{(\frac{x}{t})^{2}}{\gamma(1-(\frac{x}{t})^{2})}\right)^{\frac{1}{\gamma+1}} & -\left(\frac{\gamma}{\gamma+1}\right)^{\frac{1}{2}} < \frac{x}{t} < -\left(\frac{\gamma\epsilon^{\gamma+1}}{\gamma\epsilon^{\gamma+1}+1}\right)^{\frac{1}{2}} \\
\left(\frac{(\frac{x+R}{t})^{2}}{\gamma(1-(\frac{x+R}{t})^{2})}\right)^{\frac{1}{\gamma+1}} & \left(\frac{\gamma\epsilon^{\gamma+1}}{\gamma\epsilon^{\gamma+1}+1}\right)^{\frac{1}{2}} < \frac{x+R}{t} < \left(\frac{\gamma}{\gamma+1}\right)^{\frac{1}{2}} \\
1 & -R + \left(\frac{\gamma}{\gamma+1}\right)^{\frac{1}{2}} t < x < -\left(\frac{\gamma}{\gamma+1}\right)^{\frac{1}{2}}
\end{cases} (3.46)$$

and the expression for u as in (3.45).

We will, however, have to impose a maximum time t_c after which the solution becomes invalid owing to the information from each face meeting after this time. This time is calculated by looking at the characteristic diagram and given by $t_c = \frac{1}{2}R(\frac{\gamma+1}{\gamma})^{\frac{1}{2}}$.

We can plot the velocity to obtain a graph which is almost identical to the one for the cliff obtained earlier in Fig. 3.12. Note that the shrapnel outermost to the casing flies off with high velocity as was the case with the rock blasting.

In the cliff problem, it was assumed that the rocks were not porous. This assumption would not always hold for a cliff in general owing to the fact that rocks can be very porous. It will, however, apply well for a metal casing. Furthermore, the assumption of homogeneity in the cliff will transfer across to the metal casing. Hence our assumptions about the rock blasting fit in well with the shrapnel problem.

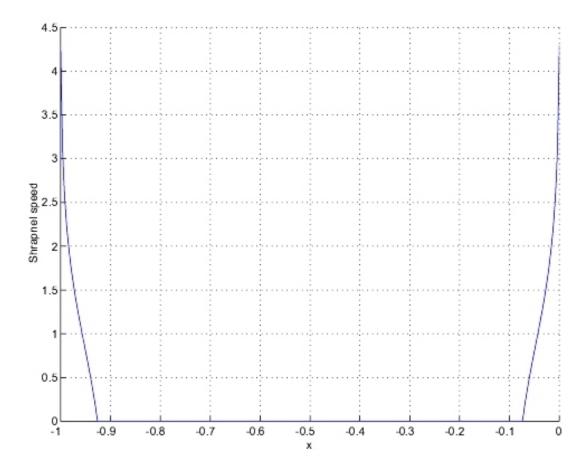


Figure 3.13: A snapshot of shrapnel velocity for a two-faced cliff with R=1.

The physical properties of the casing and charge need to be such that β (3.39) is small. Recall that β is proportional to the Young's modulus of the substance, which was of the order $10^7 Pa$ for rock. Tabulated values of Young's moduli give a value of $10^{10} Pa$ for aluminium (which is a typical casing material) and 10^{12} for steel. Thus the solution can only be justifiably applied if, in the notation of (3.39), $\frac{p_0 \rho_{c_0}}{\rho_0}$ is sufficiently small, where ρ_{c_0} is the initial casing density.

Hence we can conclude, as long as we are justified in taking the limit $\beta \to 0$, that the shrapnel velocity will mirror that of the rock velocity as in Fig. 3.12. The nondimensional parameters will change owing to the different nature of the material involved, but the rock-blasting idea gives us a good idea of how the shrapnel velocity will evolve through time.

3.4 Comparison of the models

The first models were all based on obtaining a solution for the shrapnel velocity in 1-dimension. The models considered the gas flowing into or past a thin layer of shrapnel.

Our initial idea was to model the shrapnel as a piston being pushed into a vacuum by expanding gas (section 3.1). This is quite a basic model and overestimates the piston velocity as the model does not allow for any seepage and neglects any resistance we would have acting against the piston (as we modelled it as expanding into a vacuum). Although the model is an approximation of shrapnel motion, it nevertheless gives an idea of the expected motion. It provides us with an indication that there will be a maximum velocity for the shrapnel, which is intuitively obvious, as the we don't expect the shrapnel velocity to exceed the gas velocity.

We then modelled the shrapnel as a contact discontinuity in a shock tube (section 2.2.6). This is another simple model. It, like the previous model, does not allow seepage through the discontinuity. Furthermore, it does not account for the mass of the shrapnel and so predicts a constant velocity for the shrapnel which depends on the values that p and ρ take in different regions, as demonstrated in section 2.2.6.

The aim of these basic models was to give a grounding for a model which allows gas seepage through gaps between fragments. This gave us the motivation for flow past a porous piston, in which we modelled the shrapnel fragments as a screen. For simplicity, incompressible flow was considered first (section 3.2.1). Jump conditions established in chapter 2 enable us to solve for the piston velocity analytically. As with our first model, the solution to this model indicates that the shrapnel tends to a constant velocity as $t \to \infty$, which, in this case, is the upstream gas velocity. This model is not, however, entirely realistic. This motivated the porous piston problem being modelled with compressible gas flow.

The compressible gas flow into the porous piston is the most realistic model (section 3.2.2). It is, however, also more intricate and complicated owing to the admission of shock waves. We derived a condition on the entropy jump across the piston, and showed that it is very much dependent on the drag. This affects the flow behind the porous piston (screen). We have suggested possible ideas for flow behind the screen we have to conclude that, as we do not have homentropic flow, it is nontrivial to solve for the shrapnel velocity analytically. The possibility of choked flow makes it even more difficult to put forward a soluble model as we will have to predict the nature of the downstream shock waves. However, if we make the assumption of a normal shock

as in Fig. 3.7, the model can be solved computationally. Hence, although this model is the most realistic, it is the also the most difficult to solve.

Our final model was based on cliff-blasting theory (section 3.3.2). This uses a different approach entirely. In particular, we are no longer considering gas flow past a thin layer of metal, but are considering a large section of material which consists of shrapnel and gas (equivalent to the cliff). The mathematics behind the model need certain physical assumptions about the casing to be made in order to apply the model. Assuming these assumptions are justified, the model predicts the shrapnel will fly off with high velocity from the edge of the casing, as expected. Plots of the velocity of the rock (shrapnel) as a function of position for different values of time t demonstrate this (Fig. 3.12). Hence, this model gives us some insight into shrapnel motion which the other models did not.

Chapter 4

Conclusions

4.1 Summary of work

The motivation for this problem was to develop models for shrapnel motion for use by *QinetiQ*. The background to the problem was described in chapter 1 where we outlined the general processes involved. We reinforced these ideas with experimental evidence.

We then split chapter 2 up into two parts; incompressible and compressible flow. Our aim in this chapter was to give sufficient background material to enable us to develop 1-dimensional models for shrapnel motion. Firstly, we considered background material on incompressible flows. The differences between laminar and turbulent flow were highlighted, leading to the discussion of jump conditions and conservation laws. The section on incompressible flow culminated in the concept of head losses and different loss coefficients for various flow geometries. We then established a background on compressible flow. We used some of the ideas from incompressible flow and looked in particular at choking for different pipe geometries.

In chapter 3, we used ideas from incompressible and compressible flows to develop several models for shrapnel motion. The first models were quite basic, motivating the more sophisticated later models. An alternative model which used a different approach was discussed, before bringing all the ideas together in a comparison of the models.

4.2 Further work

This dissertation has made a start on investigating shrapnel motion. Some of the points raised merit further investigation. In chapter 2, we discussed loss coefficients in some detail for incompressible flow. We were not, however, able to complete the

same analysis for compressible flow and determine loss coefficients for various pipe geometries. This needs to be looked at in more detail. A more detailed investigation of drag coefficients will also be useful. These could then be utilised in the porous piston models in both the incompressible and compressible cases.

There are also many new avenues that can be explored. Firstly, most of our modelling has been in a 1-dimensional geometry. Although this has given insight into the problem, it would be beneficial to look at casing with a spherical geometry. We would need a discussion on spherical shock waves and could perhaps look at *blast waves*, in which we would model the explosion as a sudden release of energy at the centre of the bomb at time t=0.

The screen model, although it can accommodate different sized elements in the screen (by means of the solidity) does not allow different pieces of shrapnel in the screen to have an individual acceleration, as all the elements of the screen move together as a single body. This motivates work on modelling different sizes of fragment with different accelerations, as in reality, the fragments are likely to vary in size.

Another new idea would be to develop a better two-phase model (gas and shrapnel) comparing the shrapnel to, say, so-called dusty gas flow [15].

4.3 Final comments

We have suggested possible improvements for models to the shrapnel problem. It is likely that further experimental evidence will lead to new insight into the problem, and may reinforce some of the models developed in this dissertation. We have built the foundations to the problem which should go some way into developing a more complete model describing shrapnel motion.

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